Statistics of Feynman integrals in ϕ^4-theory

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based on JHEP11 2023.160

Slides, references, data set etc. available from paulbalduf.com/research

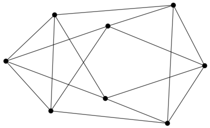
Background

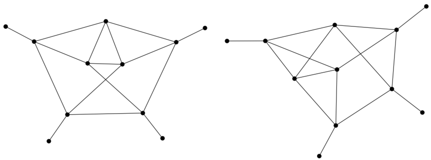
- Perturbative quantum field theory in flat (Euclidean) $D = 4 - 2\epsilon$ spacetime.
- Massless bosonic ϕ^4-theory

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{\lambda}{4!}\left(\phi^2\right)^2.$$

$\Rightarrow$ Feynman graphs have 1 type of edge, 1 type of 4-valent vertex.

- We want to understand "typical properties" of vertex-type Feynman integrals.





Periods in ϕ^4-theory

- We consider only *primitive* (=no subdivergences) vertex-type graphs G in $D = 4 - 2\epsilon$. Depend on energy scale $\ell := \ln \frac{p^2}{\mu^2}$ in a characteristic way:

$$\mathcal{F}(G) = \text{const} \cdot \left(\frac{1}{\epsilon} \frac{\mathcal{P}(G)}{L} - \mathcal{P}(G) \cdot \ell + \ell\text{-independent terms} + \mathcal{O}(\epsilon) \right)$$

- First Symanzik polynomial ψ_G. Nontrivial part of integral is the *period* [Broadhurst and Kreimer 1995; Schnetz 2010], in parametric form:

$$\mathcal{P}(G) = \left(\prod_{e \in E_G} \int_0^\infty da_e \right) \delta\left(1 - \sum_{e=1}^{|E_G|} a_e \right) \frac{1}{\psi_G^2(\{a_e\})} \in \mathbb{R}.$$

- Why consider periods?

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- Why consider periods?
  1. Prototypical "simplest possible" honest Feynman integral (no numerator, independent of renormalization scheme, independent of momenta and angles).
  2. Their sum is the primitive beta function, conjecturally dominant in MS as $L \to \infty$.

How many graphs are there?

Just generate them all and count...

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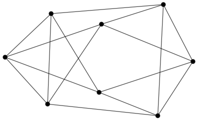
$\Rightarrow \quad > 1$ billion at 15 loops.

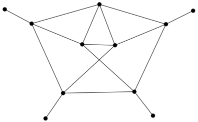
Remark: Many things are known mathematically about "large random graphs". Asymptotically, a large random 4-regular graph either contains multiedges or is primitive.

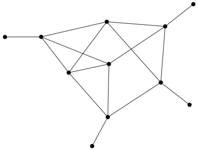
L	Primitive vertex-type graphs
3	1
4	1
5	3
6	10
7	44
8	248
9	1688
10	13094
11	114016
12	1081529
13	11048898
14	120451435
15	1393614379
16	17041643034

Symmetries

- Sometimes, the period of non-isomorphic graphs has the same value [Schnetz 2010; Panzer 2022; Hu et al. 2022].
- All "decompletions" of the same vacuum graph have the same period







- Planar dual graphs have the same period (this is rare).
- Some other symmetries.

Counts

L	All vertex-type decompletions	Vacuum graphs	planar decompletions	independent periods
3	1	1	1	1
4	1	1	1	1
5	3	2	2	1
6	10	5	5	4
7	44	14	19	9
8	248	49	58	31
9	1,688	227	235	134
10	13,094	1,354	880	819
11	114,016	9,722	3,623	6,197
12	1,081,529	81,305	14,596	55,196
13	11,048,898	755,643	60,172	543,535
14	120,451,435	7,635,677	246,573	5,769,143
15	1,393,614,379	82,698,184	1,015,339	65,117,118

$\Rightarrow$ Millions of independent integrals remain.

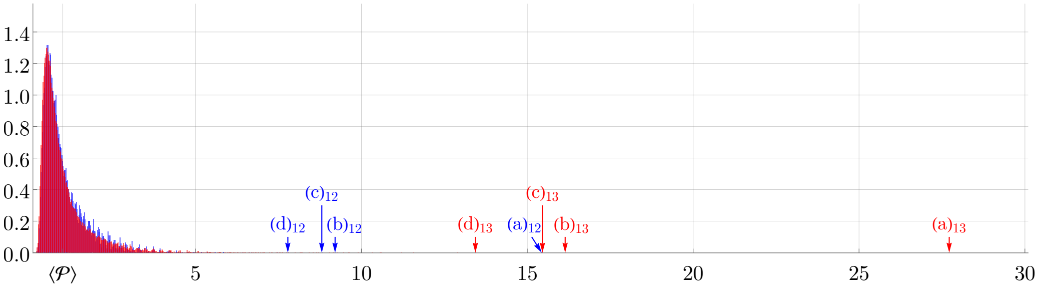
Computing periods numerically

- Periods can be quickly ($\sim$ 1h/graph) computed numerically with new algorithm up to $L \approx 16$ loops [Borinsky 2023; Borinsky, Munch, and Tellander 2023]
- Use symmetries to improve accuracy and check programs.
- Computed all graphs including 13 loops, incomplete uniform samples for $L \leq 18$. Typical accuracy 4 digits ($\approx$ 100ppm).
- $\approx 1.3 \cdot 10^6$ distinct completions (=vacuum graphs) computed, $\approx 33 \cdot 10^6$ decompletions (=vertex graphs) known.

Distribution histogram

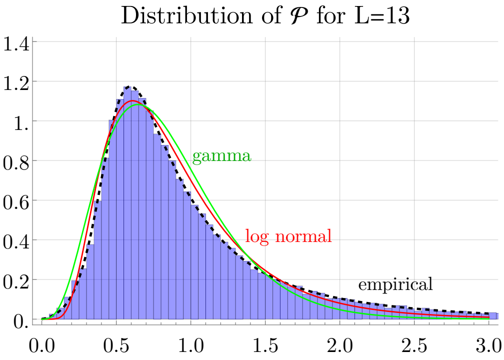
- Most periods are somewhat close to the mean $\langle \mathcal{P} \rangle$
- There are few, but extreme, outliers. Standard deviation $\sigma(\mathcal{P}) \approx \langle \mathcal{P} \rangle$.
- The pattern of outliers repeats at each loop order, but scaled.

Distribution of periods, without symmetry factor, normalized to unit mean



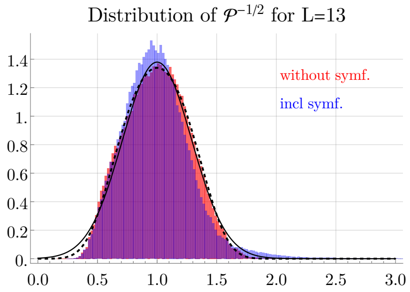
Continuous part of the distribution

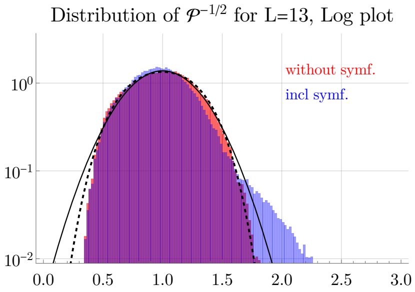
- Distribution of *uniform* (=period value multiplied with symmetry factor, not period sampled proportional to symmetry factor) periods is none of the usual well-known distribution functions.
- Can be modeled empirically with 5 free parameters.
- Distribution is essentially unchanged at higher loop order.



A curious observation

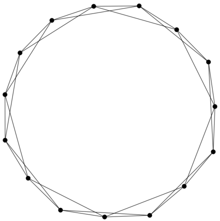
- The quantity $\frac{1}{\sqrt{\mathcal{P}}}$ is almost normally distributed.
- I don't know why.





Which ones are the outliers?

- The Zigzag graphs (=$(1,2)$-circulants) and their friends.
- They look "symmetric", but that's deceptive, overall only weak correlation between $\mathcal{P}$ and symmetry factor.



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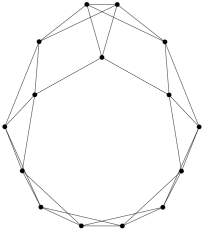
$\mathcal{P} \approx 832206.8$ $|\mathrm{Aut}| = 30$



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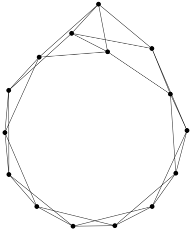
$\mathcal{P} \approx 484645.5$ $|\mathrm{Aut}| = 4$



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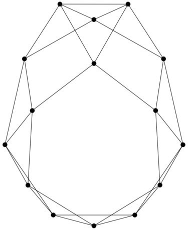
$\mathcal{P} \approx 464116.5$ $|\mathrm{Aut}| = 2$



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$\mathcal{P} \approx 403425.9$ $|\mathrm{Aut}| = 2$

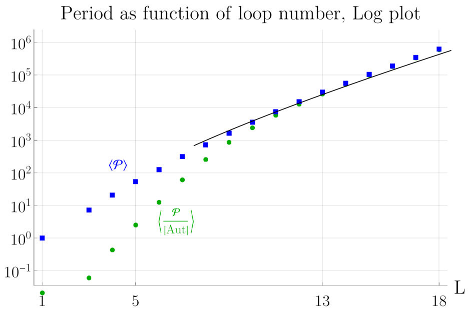
largest $\longrightarrow$ 4$^{\text{th}}$ largest

Behavior of the mean

- Leading asymptotic growth of full beta function in MS from instanton calculation [A. J. McKane, Wallace, and Bonfim 1984; Alan J. McKane 2019] + conjecture that primitive graphs dominate MS + asymptotics of number of graphs [Cvitanović, Lautrup, and Pearson 1978; Borinsky 2017] implies

$$\langle \mathcal{P} \rangle \sim C \cdot \left(\frac{3}{2}\right)^{L+3} L^{\frac{5}{2}}.$$

- Matches observed growth, potentially with different constant C.



Conclusion

- There are very many Feynman graphs. Obtained counts and asymptotics for numbers, symmetry factors, planarity etc.

Conclusion

- There are very many Feynman graphs. Obtained counts and asymptotics for numbers, symmetry factors, planarity etc.
- The distribution is largely "smooth" apart from a few extreme outlier graphs.
- Obtained various averages, standard deviations, higher moments, correlation coefficients.
- Obtained numerical approximation of 18-loop $O(N)$-symmetric primitive ϕ^4 beta function.

What's that good for?

- Beta function $\Rightarrow$ critical exponents for all systems in the same universality class.
- General understanding of perturbative QFT and divergence of perturbation series.
- Understand accuracy of samples, or of extrapolation of special classes (e.g. planar graphs are not a good proxy for all graphs).
- Use empirical data for weighted Monte-Carlo sampling of graphs, overcome the problem of large variance (see my talk at Theory Canada / my website).

Thank you!

for staying for the last talk :-)

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