William Tutte Colloquium

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Graph theory and Feynman integrals

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Based on JHEP 11(2023), 160; arXiv 2403.16217; work in progress.

Slides, related papers, data set etc. available from paulbalduf.com/research

What is a free field theory?

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- Naive intuition: The field will assume a state of "minimum energy". This can't be because energy is conserved. Rather: The field will assume a state of stationary (often minimum) *action*. Action density (=Lagranian) in the simplest case:

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$$\mathcal{L} = \frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\partial_x \phi)^2.$$

- Stationary action means $0 = \delta \int \mathcal{L} \, \mathrm{d}t \mathrm{d}x$, or

$$0 = \delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} - \frac{\partial}{\partial x} \frac{\partial \mathcal{L}}{\partial(\partial_x \phi)}$$
$$\Rightarrow \left(\partial_t^2 - \partial_x^2\right)\phi = 0.$$

- This *equation of motion* is linear in ϕ, hence it is a *free* field: Sums of solutions $\phi_1(x) + \phi_2(x)$ are solutions, too. They do not influence each other.

What is an interacting field theory?

- An action density that is *quadratic* in the field ϕ gives rise to *linear* equations of motion and hence a *free* field.
- As soon as $\mathcal{L}$ is more than quadratic, the equation of motion becomes non-linear. $\Rightarrow$ sums of solutions are not (necessarily) a solution any more, they interact. For example, energy of 2 waves is no longer the sum of individual energies.

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- We will only ever consider one type of interaction:

$$\mathcal{L} = \frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\partial_x \phi)^2 - \frac{\lambda}{4!}\phi^4.$$

When interpreted as a quantum field theory in 3-dimensional space (+ time), this is called ϕ^4*-theory*.

Quantum field theory

- So far we considered classical field theory, where $\delta\mathcal{L} = 0$. In quantum field theory, instead all field configurations are weighted with $\exp\left(\frac{i}{\hbar}\int \mathcal{L}\, \mathrm{d}t\, \mathrm{d}x\right)$, where $\hbar$ is the Planck constant (=quantum of action).
- Quantum field is operator-valued. We no longer look for a solution "function" $\phi(x)$, but instead, for correlation functions $\langle \phi(x_1)\phi(x_2)\cdots\phi(x_k)\rangle$.

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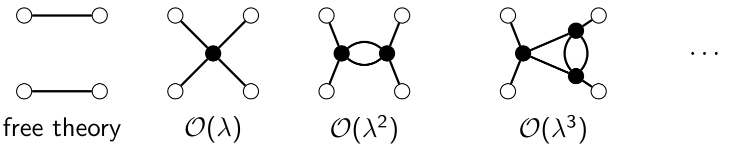
- Can compute correlation functions of the free QFT analytically in terms of *propagators*, time-ordered 2-point functions $\langle T\phi(x)\phi(y)\rangle = G_F(x,y)$.
- For the interacting theory, use *perturbation theory*: expand the exponential as a power series in the parameter λ of the $\frac{\lambda}{4!}\phi^4$ interaction term.
- Notice the interaction term is *local*, it involves four factors of the field at the same point. Hence, one power of λ amounts to interaction at one single point.

Quantum field theory

- In quantum field theory, all field configurations are weighted with $\exp\left(\frac{i}{\hbar}\int \mathcal{L}\, \mathrm{d}t\, \mathrm{d}x\right)$.
- *Propagator*: time-ordered 2-point functions of free field $\langle T\phi(x)\phi(y)\rangle = G_F(x,y)$.
- *Perturbation theory*: expand the exponential as a power series in the parameter λ of the $\frac{\lambda}{4!}\phi^4$ interaction term.
- The interaction term is *local*, and it involves four factors of the field. Hence, one power of λ amounts to interaction at one single point.
- This situation can be depicted with *Feynman graphs* [Feynman 1949; Feynman 1948]. Every interaction is a 4-valent vertex, every propagator is an edge. E.g. 4-point correlation function:

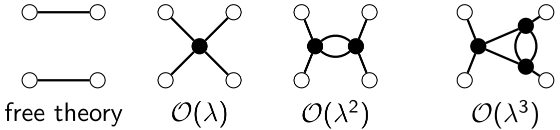


(each of them comes in multiple different orientations not shown here)

- Feynman graphs are graduated by "loop number" $L \geq 0$ (=dimension of cycle space).

Feynman integrals

- The series expansion of a correlation function in quantum field theory can be organized in terms of *Feynman graphs*, e.g. the 4-point function



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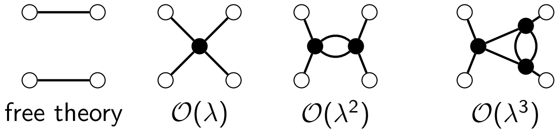

- The interaction can happen *anywhere* $\Rightarrow$ we need to integrate the position of each interaction vertex over the whole spacetime.
- Typical experiments in QFT measure *scattering*, not spacial correlation. One typically wants to compute a Fourier transform, i.e. Feynman integrals as functions of momenta, not of positions. Every linearly independent cycle ("loop") gives one integral over undetermined momenta. However, we will use *parametric* integrals, where every edge gives one integral.

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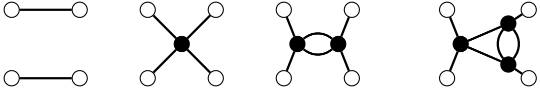

free theory $\mathcal{O}(\lambda)$ $\mathcal{O}(\lambda^2)$ $\mathcal{O}(\lambda^3)$

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- Typical experiments in QFT measure *scattering*, not spacial correlation. One typically wants to compute a Fourier transform, i.e. Feynman integrals as functions of momenta, not of positions. Every linearly independent cycle ("loop") gives one integral over undetermined momenta. However, we will use *parametric* integrals, where every edge gives one integral.
- Generally, we need the sum over *all* possible graphs, not just individual graphs.
- Goal of this talk: **Understand the "typical" properties of Feynman integrals.**

Periods in ϕ^4-theory

- A Feynman integral depends on the momenta of all its external vertices, on masses of particles, on conventions regarding normalization etc.
- For this talk, avoid renormalization issues and concentrate on the "simplest" case: Vertex-type graphs without 2-valent or 4-valent subgraphs (i.e. cyclically 6-edge connected graphs).
  These graphs are *primitive* in the Hopf algebra of Feynman graphs [Kreimer 1998].

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  These graphs are *primitive* in the Hopf algebra of Feynman graphs [Kreimer 1998].
- We ignore the dependence on masses and angles, and only ask for the dependence on the overall energy *scale* $\ell := \frac{p^2}{\mu^2}$, where μ is an arbitrary reference momentum. These (renormalized) integrals of a primitive vertex-type graph G have the form

$$\mathcal{F}(G) = -\text{const} \cdot \mathcal{P}(G) \cdot \ell \quad (+\ \ell - \text{independent terms}).$$

- The non-trivial coefficient $\mathcal{P}(G)$ is called the *period* of the graph G [Broadhurst and Kreimer 1995; Schnetz 2010]. It is a finite real number.

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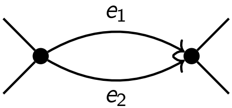
- The non-trivial coefficient $\mathcal{P}(G)$ is called the *period* of the graph G [Broadhurst and Kreimer 1995; Schnetz 2010]. It is a finite real number.
- $\mathcal{P}(G)$ is a particular type of Feynman integral. Can be expressed in many forms, for example with the first Symanzik polynomial ψ_G (=dual Kirchhoff polynomial):

$$\mathcal{P}(G) = \left(\prod_{e \in E_G} \int_0^\infty \mathrm{d}a_e \right) \delta\left(1 - \sum_{e=1}^{|E_G|} a_e \right) \frac{1}{\psi_G^2(\{a_e\})} \in \mathbb{R}.$$

- **In this talk, periods of ϕ^4-theory are the only type of Feynman integral to be considered.**

Example: loop order 1

- Smalles nontrivial vertex-type graph: $G =$ 

  (This is the only non-simple graph we will ever consider)
- Symanzik polynomial is the sum over spanning trees. Here, trees have only one edge, $T_1 = \{e_1\}$ and $T_2 = \{e_2\}$.

$$\psi_G = \sum_{T \text{spanning}} \prod_{e \notin T} a_e = a_2 + a_1 .$$

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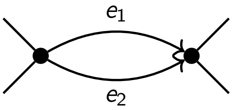

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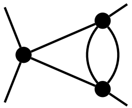
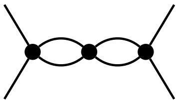
- Period is

$$\mathcal{P}(G) = \int_0^\infty \mathrm{d}a_1 \int_0^\infty \mathrm{d}a_2 \, \delta(1 - a_1 - a_2) \frac{1}{(a_2 + a_1)^2} = \int_0^1 \mathrm{d}a_1 \, \frac{1}{((1 - a_1) + a_1)^2} = 1.$$

Example: loop order 2

- At loop order 2, graphs have 3 vertices, 4 internal edges and 4 external edges. Two examples:





Example: loop order 2

- At loop order 2, graphs have 3 vertices, 4 internal edges and 4 external edges. Two examples:

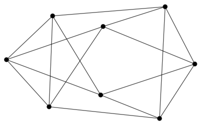
- No such graph is cyclically 6-edge connected. There are no primitives at $L = 2$.
- In fact, no graph that contains a multi edge as a subgraph is cyclically 6-edge connected $\Rightarrow$ for $L > 1$, all primitive graphs are simple.

Completions and decompletions

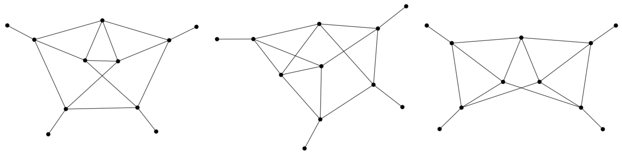
- The period $\mathcal{P}(G)$ is defined for a vertex-type graph, i.e. G has 4 external edges. At L loops, G has $L+1$ (internal) vertices.

Completions and decompletions

- The period $\mathcal{P}(G)$ is defined for a vertex-type graph, i.e. G has 4 external edges. At L loops, G has $L+1$ (internal) vertices.
- Merge the 4 external edges to a new vertex. The resulting graph has $L+2$ vertices and no external edges. It is called the *completion* of G. Loop number or completion refers to L of decompletion.



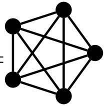
(completion)

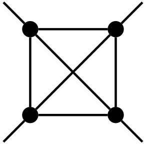


(3 non isomorphic decompletions)

- Every vertex-type G has a unique completion, but a single completion can have multiple non-isomorphic decompletions.
- All decompletions G of some fixed completion have the same period $\mathcal{P}(G)$. We will therefore often work with completions.

Example: loop order 3

- We saw: There are no primitive graphs with $L = 2$.
- There is exactly one primitive graph on $L + 2 = 5$ vertices, $K_5 =$ 
- All vertices in K_5 are equivalent $\Rightarrow$ removing any of them gives the same decompletion

$G =$ 

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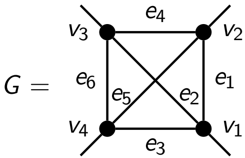

- Enumerating spanning trees is tedious. Better: Use matrix-tree theorem. Labelled graph Laplacian:

$$\tilde{\mathbb{L}} = \begin{pmatrix} -a_1 - a_2 - a_3 & a_1 & a_2 & a_3 \\ a_1 & -a_1 - a_4 - a_5 & a_4 & a_5 \\ a_2 & a_4 & -a_2 - a_4 - a_6 & a_6 \\ a_3 & a_5 & a_6 & -a_3 - a_5 - a_6 \end{pmatrix}$$

- Leave out any one row and column to obtain reduced Laplacian $\mathbb{L}$. Then the Kirchhoff polynomial is

$$\tilde{\psi}_G = \det(\mathbb{L}) = a_1(a_2 + a_4)(a_3 + a_5) + a_1(a_2 + a_3 + a_4 + a_5)a_6 + \ldots$$

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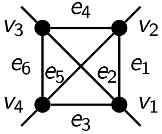
$G =$ 

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- Symanzik polynomial $\psi(\{a_e\}) = \tilde{\psi}\left(\left\{\frac{1}{a_e}\right\}\right)\prod_{e\in E_G} a_e$

$$\psi_G = a_1(a_2+a_3)(a_4+a_5) + a_4a_5a_6 + a_1(a_2+a_3+a_4+a_5)a_6 + a_2a_5(a_4+a_6) + a_3a_4(a_5+a_6) + a_2a_3(a_4+a_5+a_6)$$

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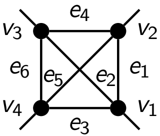
$G =$ 

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- Period integral can be solved with some effort,

$$\mathcal{P}(G) = \left(\prod_{e \in E_G} \int_0^\infty \mathrm{d}a_e \right) \delta\left(1 - \sum_{e=1}^{|E_G|} a_e\right) \frac{1}{\psi_G^2(\{a_e\})} = 6\zeta(3) \approx 7.212$$

Summary of physics motivation and definitions

- Feynman graphs are a graphical notation for Feynman integrals. They appear in the perturbative series-solution in interacing field theories.
- We concentrate on the special case of *primitive vertex-type graphs in ϕ^4-theory*, these graphs are 4-regular except for 4 external edges, and cyclically 6-edge connected.
- The Feynman integral of such a graph G is a real number, the *period*

$$\mathcal{P}(G) := \left(\prod_{e \in E_G} \int_0^\infty \mathrm{d}a_e \right) \delta\left(1 - \sum_{e=1}^{|E_G|} a_e\right) \frac{1}{\psi_G^2(\{a_e\})} \in \mathbb{R}.$$

- The sum of all periods determines the scale-dependence of the theory, i.e. what happens if all momenta and all masses of a theory are scaled by the same factor.
- The period integrand can be computed algorithmically from the Laplacian, but the integral is hard to solve even for small graphs.
- Goal: **Understand "typical" properties of periods and their underlying graphs.**

Theory with O(N)-symmetry

- So far we have considered a *scalar* theory, i.e. $\phi(x) \in \mathbb{R}$ is a number.
- Relatively easy to generalize to *vector* theory where $\vec{\phi}(x) \in O(N)$ is a N-component vector. Orthogonal group $O(N)$ means that the theory is invariant under rotations of this vector.
- Interaction term $\frac{\lambda}{4!}\phi^4$ becomes $\frac{\lambda}{4!}(\vec{\phi} \cdot \vec{\phi})^2$.

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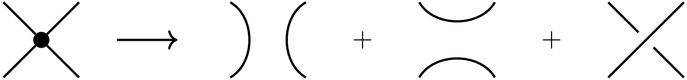
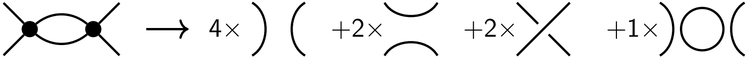
- Interaction term $\frac{\lambda}{4!}\phi^4$ becomes $\frac{\lambda}{4!}(\vec{\phi}\cdot\vec{\phi})^2$.
- In terms of indices: $\vec{\phi}\cdot\vec{\phi} = \sum_j \phi_j \phi_j$. Graphically: Pairs of indices are connected, the 4 edges of a vertex can have at most 2 distinct indices. The vertex is the sum of 3 possible ways to connect:



- For the entire graph: Sum over all such "vertex decompositions". Obtain collection of open and closed lines, exactly $3^{|V_G|}$ terms.
- Every closed line is a trace $\sum_j \mathbb{1}_{j,j}$. Vector has N components $\Rightarrow$ every closed line gives a factor N.

O(N) symmetry for vertex-type graphs

- Sum over all vertex decompositions. Obtain collection of open and closed lines. Every closed line gives a factor of N.
- If the original graph was of vertex-type (=4 external edges), we obtain decompositions with 4 external edges and possibly circuits.



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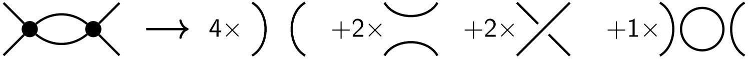
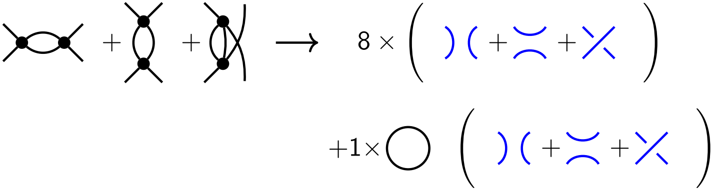


- Summing over all orientations of a vertex-type graph, the sum of decompositions is proportional to the decomposition of a single vertex.



Circuit partition polynomial

- Every circuit in a decomposition gives a summand N. Summing over all orientations of a vertex-type graph G, the sum over all decompositions is proportional to that of a single vertex.
- The constant of proportionality is called *Circuit partition polynomial*

$$J(G,N) := \sum_{\substack{\text{decompositions}\\ \text{of vertices in } G}} N^{\#\text{circuits in decomposition}}$$

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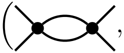
- For example:

$J\Big($  $, N\Big) = N + 8$

- In the parturbation series, every Feynman integral $\mathcal{F}(G)$ obtains a factor $J(G,N)$.

More about $O(N)$ symmetry later...

How many graphs are there?

Just generate them all and count...

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Just generate them all and count...

Number grows factorially.
> 1 billion at 15 loops.

L	Primitive decompletions
1	1
2	0
3	1
4	1
5	3
6	10
7	44
8	248
9	1,688
10	13,094
11	114,016
12	1,081,529
13	11,048,898
14	120,451,435
15	1,393,614,379
16	17,041,643,034

Period Symmetries

- Know already: All decompletions of the same completion have the same period.

Period Symmetries

- Know already: All decompletions of the same completion have the same period.
- There are a few other *symmetries*, where the period of non-isomorphic graphs has the same value [Schnetz 2010; Panzer 2022; Hu et al. 2022].
    - Planar dual graphs have the same period.
    - In a 3-vertex cut, the period is the product of the two sides' periods
    - In a 4-vertex cut, can take the planar dual on either side or "twist" the connection at the cut vertices.
- Write another program to try all of these and find symmetric graphs.

Counts of primitive graphs

L	Vertex-type graphs "decompletions"	Vacuum graphs "completions"	planar decompletions	independent periods
3	1	1	1	1
4	1	1	1	1
5	3	2	2	1
6	10	5	5	4
7	44	14	19	9
8	248	49	58	31
9	1,688	227	235	134
10	13,094	1,354	880	819
11	114,016	9,722	3,623	6,197
12	1,081,529	81,305	14,596	55,196
13	11,048,898	755,643	60,172	543,535
14	120,451,435	7,635,677	246,573	5,769,143
15	1,393,614,379	82,698,184	1,015,339	65,117,118

$\Rightarrow$ Exploiting all symmetries, still millions of independent period integrals remain.

Relation to random graphs on $n = L + 2$ vertices

- Our *primitive completions* are 4-regular and cyclically 6-edge connected graphs.
- Consider space of random 4-regular simple graphs $\mathcal{G}_{n,4}$ on $n = L + 2$ vertices.
- Known for $G \in \mathcal{G}_{n,4}$ in the limit $n \to \infty$:
  - G is almost surely 4-edge connected [Wormald 1981],
  - asymptotic number of graphs in $\mathcal{G}_{n,4}$ coincides with asymptotic number of primitive graphs [Bender and Canfield 1978; Bollobás 1982; Borinsky 2017],
  - $\langle |\mathrm{Aut}(G)| \rangle \to 1$ [McKay and Wormald 1984], ...

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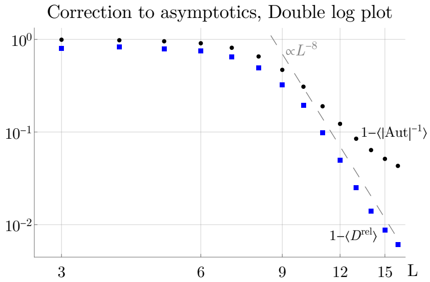
  - $\langle |\mathrm{Aut}(G)| \rangle \to 1$ [McKay and Wormald 1984], . . .
- $\Rightarrow$ Expectation: To leading asymptotic order, $\mathcal{G}_{n,4}$ is a good model for our graphs.
- We deal with graphs on $n \leq 20$ vertices. Are we in the asymptotic region?

Are we in the $n \to \infty$ asymptotic region for $\mathcal{G}_{n,4}$?

Symmetry factors $\langle |\mathrm{Aut}(G)| \rangle \to 1$?
[McKay and Wormald 1984]

All decompletions non-isomorphic, $\left\langle D^{\mathrm{rel}} \right\rangle \to 1$?



$\Rightarrow$ asymptotic domain starts at $n \sim 10$.

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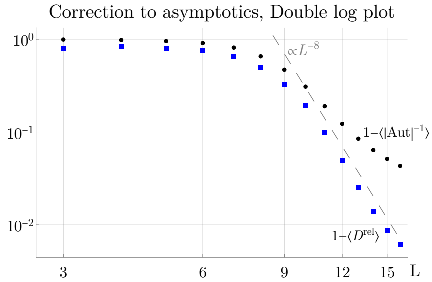
All decompletions non-isomorphic, $\langle D^{\mathrm{rel}} \rangle \to 1$?

Cycles of length k are Poisson distributed with mean $\tilde{X}_k := \frac{3^k}{2k}$? [Bollobás 1980; McKay, Wormald, and Wysocka 2004]



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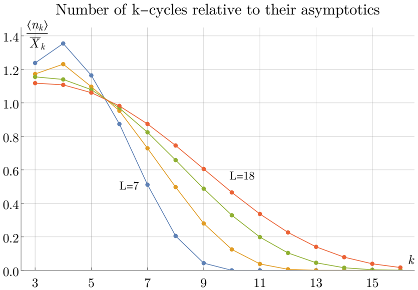


$\Rightarrow$ only good for short cycles where $k \ll n$.

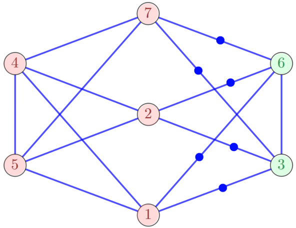
$\Rightarrow$ whether we are in the asymptotic domain depends on the details.

Invariants

- We have seen that $\mathcal{P}$ is invariant under several symmetry operations on the graph. Which other quantities are invariant?

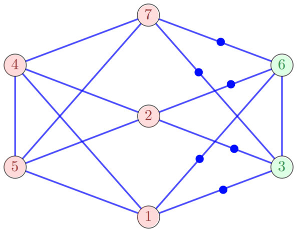
Cut invariants

- A vertex-induced cut $C \subseteq E_G$ separates the graph into exactly 2 connected components. Let c_j number of j-edge cuts.



Cut invariants

- A vertex-induced cut $C \subseteq E_G$ separates the graph into exactly 2 connected components. Let c_j number of j-edge cuts.
- For primitive graphs: Only even j possible, and $c_2 = 0$ and $c_4 = |V_G|$.
- c_6 is a period invariant [Panzer unpublished].
- c_8 can be made invariant with certain corrections, potentially c_{10}, too [B, Panzer, Yeats in progress].



Hepp bound

- *Hepp bound* $\mathcal{H}(G)$ [Hepp 1966; Panzer 2022] arises from "tropicalization" of period integral, replacing Symanzik polynomial ψ_G.

$$\mathcal{H}(G) := \left(\prod_{e \in E_G} \int_0^\infty \mathrm{d}a_e \right) \delta\left(1 - \sum_{e=1}^{|E_G|} a_e \right) \frac{1}{\psi^2_{G,\mathrm{trop}}(\{a_e\})} \in \mathbb{Q}$$

$\psi_{G,\mathrm{trop}} :=$ maximum monomial of ψ_G.

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$$\psi_{G,\text{trop}} := \text{maximum monomial of } \psi_G.$$

- Which monomial is maximum depends on the particular values of edge variables $\{a_e\}$. Orderings of edge variables $\leftrightarrow$ maximum monomials $\leftrightarrow$ spanning trees.
- Each ordering of $\{a_e\}$ is called *Hepp sector*. $\mathcal{P}(G)$ is the sum of all Hepp sector integrals.
- The Hepp sector integrals of $\mathcal{H}(G)$ are over monomials $\Rightarrow$ can be done analytically $\Rightarrow$ recursive combinatorial formula for $\mathcal{H}(G)$ without explicit integration.

Martin invariant

- Recall the Circuit partition polynomial $J(G,N)$ for $O(N)$-valued fields.
- If G is a decompletion, the *Martin invariant* is the linear coefficient of the Martin polynomial[1] $M(G,N) := \frac{J(G,N-2)}{N-2}$ [Panzer and Yeats 2023].
- Linear coefficient = derivative, evaluated at zero $\Rightarrow$ the value $N=-2$ of the O(N)-symmetric field is combinatorially interesting.

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- Linear coefficient = derivative, evaluated at zero ⇒ the value $N=-2$ of the O(N)-symmetric field is combinatorially interesting.
- Replace every edge in G by k parallel edges, compute J and M, obtain "higher" Martin invariant $M^{[k]}$.
- Like Hepp bound, can be computed by explicit combinatorial enumeration without any integral

[1] trivial factors omitted.

Summary: Counts, symmetries, invariants

- There are very many Feynman graphs. Their number grows factorially.
- $\mathcal{G}_{n,4}$ is a reasonably good model for primitive graphs for $n > 10$.
- Not all non-isomorphic graphs have distinct periods, some are related by *period symmetries*. Most importantly, completion symmetry.
- There are *period invariants* (other than the period itself) which respect the same symmetries. We mentioned three: The number of 6- and 8-edge cuts, the Hepp bound and the Martin invariant.
- The invariants are much easier to compute than the period itself.

Computing periods numerically

- Periods can be quickly ($\sim$ 1h/graph) computed numerically with new algorithm up to $L \approx 16$ loops [Borinsky 2023; Borinsky, Munch, and Tellander 2023]
- Use symmetries to improve accuracy and check programs. Use product symmetry to obtain more periods.

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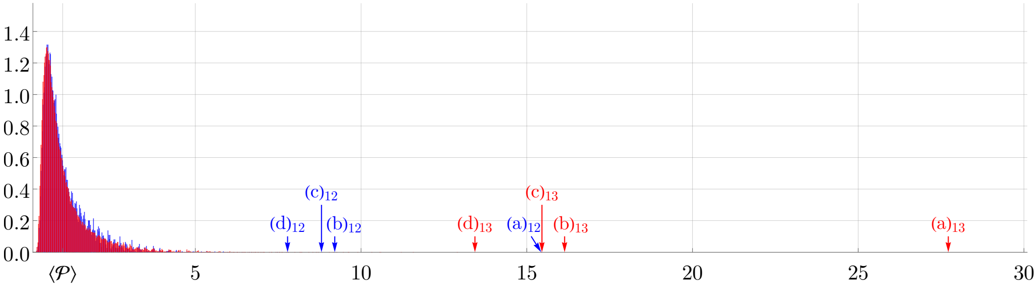
- Use symmetries to improve accuracy and check programs. Use product symmetry to obtain more periods.
- Computed all graphs including 13 loops, incomplete samples for $L \leq 18$. Typical accuracy 4 digits ($\approx$ 100ppm).
- $\approx 2 \cdot 10^6$ distinct completions (=vacuum graphs) computed, $\approx 33 \cdot 10^6$ decompletions (=vertex graphs) known.

Distribution of periods

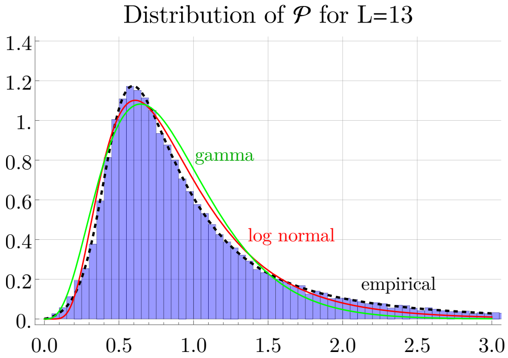
- Most periods are somewhat close to the mean $\langle \mathcal{P} \rangle$
- There are few, but extreme, outliers. Standard deviation $\sigma(\mathcal{P}) \approx 100\% \langle \mathcal{P} \rangle$.
- The pattern of outliers repeats at each loop order, but scaled.

Distribution of periods, without symmetry factor, normalized to unit mean



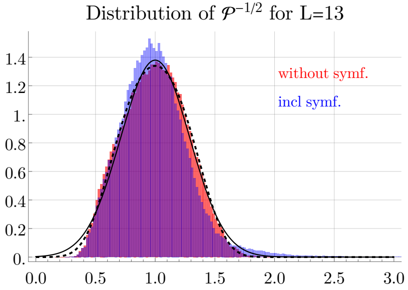
Continuous part of the distribution

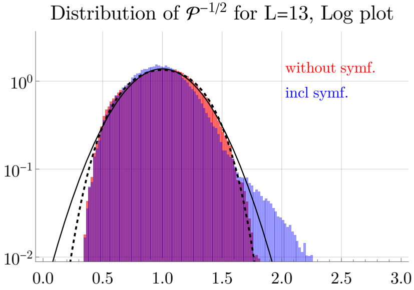
- Distribution of (uniformly sampled) periods is none of the usual well-known distribution functions.
- Can be modeled empirically with 5 free parameters.
- Shape of distribution is essentially unchanged at higher loop order, just scaled.



A curious observation

- The quantity $\frac{1}{\sqrt{\mathcal{P}}}$ is almost normal distributed.
- I don't know why.



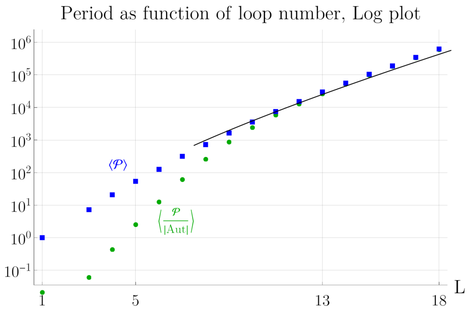


Behavior of the mean

- Leading asymptotic growth of full beta function in MS from instanton calculation [McKane, Wallace, and Bonfim 1984; McKane 2019] + conjecture that primitive graphs dominate MS + asymptotics of number of graphs [Cvitanović, Lautrup, and Pearson 1978; Borinsky 2017] implies

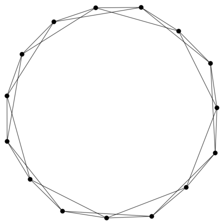
$$\langle \mathcal{P} \rangle \sim C \cdot \left(\frac{3}{2}\right)^{L+3} L^{\frac{5}{2}}.$$

- Matches observed growth, potentially with different constant C.



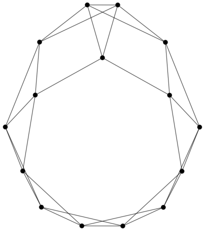
Which ones are the outliers?

- The zigzag graphs $(= (1,2)$-circulants) and their cousins.
- They look "symmetric", but that's deceptive, overall only weak correlation between $\mathcal{P}$ and symmetry factor.



/bcde/2346/345/67/58/8a/79/9c/ab/ce/bd/d/e/e//

$\mathcal{P} \approx 832206.8$ $\quad |\mathrm{Aut}| = 30$



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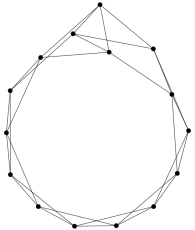
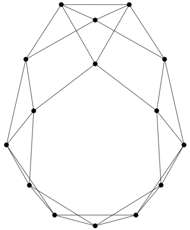
$\mathcal{P} \approx 484645.5$ $\quad |\mathrm{Aut}| = 4$



/bcde/2359/345/47/68/9a/78d/8b/b/ac/ce/d/e/e//

$\mathcal{P} \approx 464116.5$ $\quad |\mathrm{Aut}| = 2$



/bcde/2467/347/578/69/8ac/9d/b/ab/de/bc//e/e//

$\mathcal{P} \approx 403425.9$ $\quad |\mathrm{Aut}| = 2$

largest $\longrightarrow$ 4$^{\text{th}}$ largest

Correlations

- There are very many periods at large loop order, it takes long to numerically compute all of them.
- Can we guess their value with some simple function $\bar{\mathcal{P}}(G)$?

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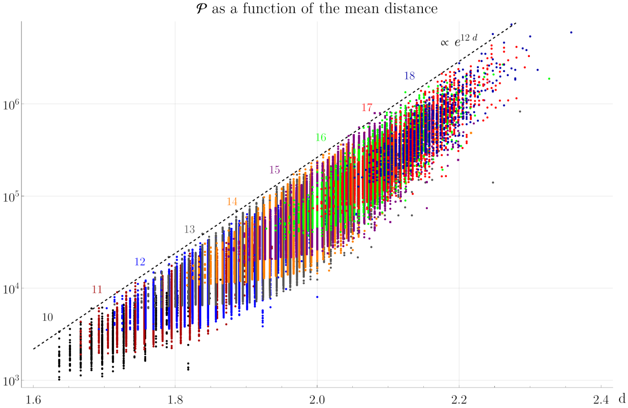
- There are very many periods at large loop order, it takes long to numerically compute all of them.
- Can we guess their value with some simple function $\bar{\mathcal{P}}(G)$?
- The period of a graph G is correlated with many properties of G, we examined ~ 150 distinct properties empirically.
- Measure of reached accuracy: relative standard deviation of the ratio $\mathcal{P}/\bar{\mathcal{P}}$,

$$\delta := \frac{\sigma\left(\frac{\mathcal{P}}{\bar{\mathcal{P}}}\right)}{\left\langle \frac{\mathcal{P}}{\bar{\mathcal{P}}} \right\rangle} = \sigma\left(\frac{\frac{\mathcal{P}}{\bar{\mathcal{P}}} - \left\langle \frac{\mathcal{P}}{\bar{\mathcal{P}}} \right\rangle}{\left\langle \frac{\mathcal{P}}{\bar{\mathcal{P}}} \right\rangle} \right).$$

- Recall: At fixed L, all 4-regular graphs have the same number of edges and vertices.

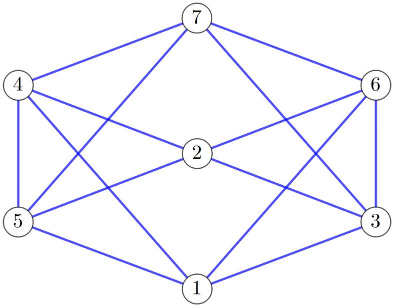
Average vertex distance

- Average of the shortest path between all pairs of vertices, where each edge has length 1.
- Relatively fast to count, clearly correlated, but low accuracy $\delta \approx 30\%$.
- Note that the graphs of different loop order align. This correlation is universal across all loop orders.



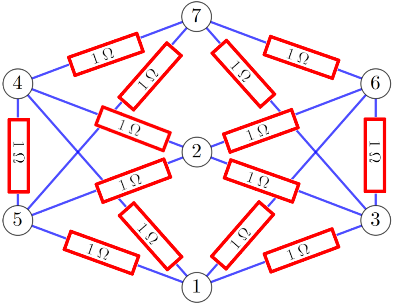
Average resistance (Kirchhoff index)

- Consider a completion G.



Average resistance (Kirchhoff index)

- Consider a completion G.
- Assign unit electrical resistance to every edge.

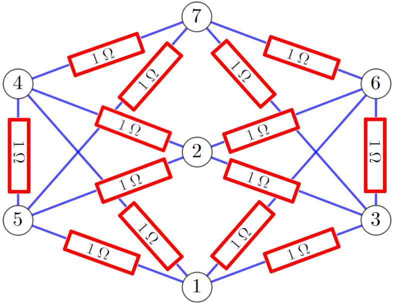


Average resistance (Kirchhoff index)

- Consider a completion G.
- Assign unit electrical resistance to every edge.
- Resistance r_{v_i,v_j} between vertices v_i and v_j. Matrix of resistances can be computed from the pseudoinverse $\mathbb{L}^+$ of the (unlabelled) Laplacian $\mathbb{L}$,

$$r_{i,j} = \mathbb{L}^+_{i,i} + \mathbb{L}^+_{j,j} - \mathbb{L}^+_{i,j} - \mathbb{L}^+_{j,i}.$$

- *Kirchhoff index* $R(G)$ = average resistance

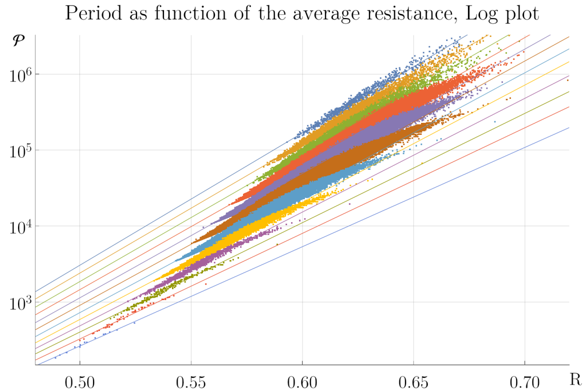


from vertex ... 1 2 3 4 5 6 7

to vertex ... 1 2 3 4 5 6 7

$$\begin{pmatrix} 0 & \frac{1}{2} & \frac{29}{30} & \frac{29}{30} & \frac{29}{30} & \frac{29}{30} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{29}{30} & \frac{29}{30} & \frac{29}{30} & \frac{29}{30} & \frac{1}{2} \\ \frac{29}{30} & \frac{29}{30} & 0 & \frac{8}{15} & \frac{8}{15} & \frac{2}{3} & \frac{29}{30} \\ \frac{29}{30} & \frac{29}{30} & \frac{8}{15} & 0 & \frac{2}{3} & \frac{8}{15} & \frac{29}{30} \\ \frac{29}{30} & \frac{29}{30} & \frac{8}{15} & \frac{2}{3} & 0 & \frac{8}{15} & \frac{29}{30} \\ \frac{29}{30} & \frac{29}{30} & \frac{2}{3} & \frac{8}{15} & \frac{8}{15} & 0 & \frac{29}{30} \\ \frac{1}{2} & \frac{1}{2} & \frac{29}{30} & \frac{29}{30} & \frac{29}{30} & \frac{29}{30} & 0 \end{pmatrix} \Rightarrow R(G) = \frac{71}{90}$$

Average resistance

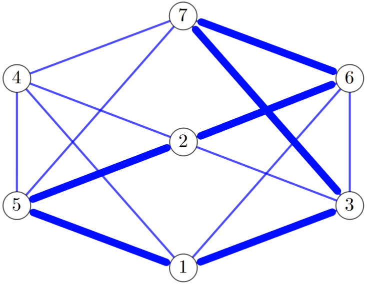


(Colors = loop orders 5 . . . 18)

- Resistance approximation reaches $\delta \approx 5\%$ with linear fit.
- Nice interpretation: spatially “larger” (=less dense) graphs have larger periods.

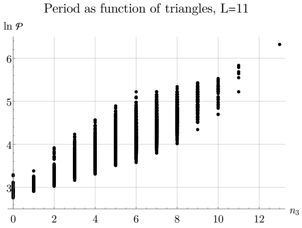
Cycles

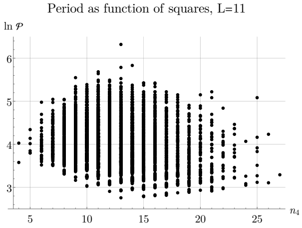
- $n_j(G)$ the total number of cycles of length $j \in \mathbb{N}$ contained in a graph G.
- Do not have to be mutually disjoint, or visit all vertices.
- For simple graphs: $n_1 = n_2 = 0$, n_3= number of triangles, n_4=number of squares, ...



Cycles

- $\mathcal{P}$ is somewhat correlated with n_3, and almost uncorrelated with $n_{j\geq 4}$.

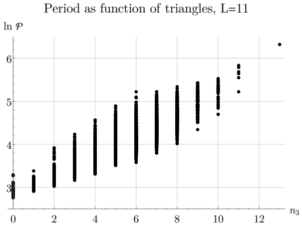


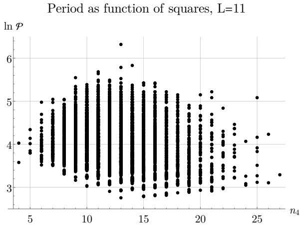


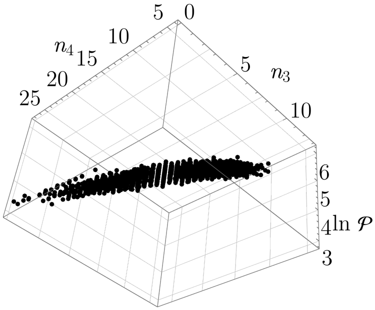
Cycles

- $\mathcal{P}$ is somewhat correlated with n_3, and almost uncorrelated with $n_{j\geq 4}$.
- n_3 and n_4 together are good!
- $\Rightarrow$ Multi-linear ansatz, scaled to the known asymptotic cycle count $\tilde{X}_j = \frac{3^j}{2j}$ from $\mathcal{G}_{n,4}$:

$$\ln \bar{\mathcal{P}} := f_0 + \sum_{j=3}^{j_{\max}} f_j \frac{2j \cdot n_j}{3^j}.$$







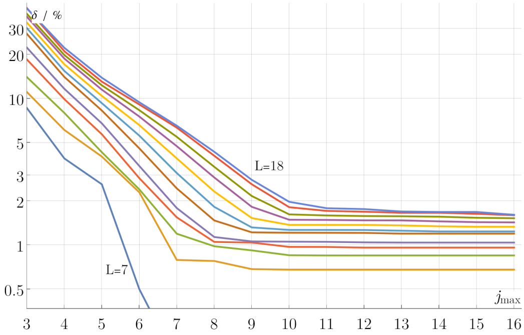
Cycles

- Multi-linear function of j-cycle count n_j,

$$\ln \bar{\mathcal{P}} := f_0 + \sum_{j=3}^{j_{\max}} f_j \frac{2j \cdot n_j}{3^j}.$$

- Approximation gets better with increasing $j_{\max}$. Saturated at $j \approx 10$, accuracy $\delta \approx 2\%$.

Relative standard deviation as function of $j_{\max}$, Log plot



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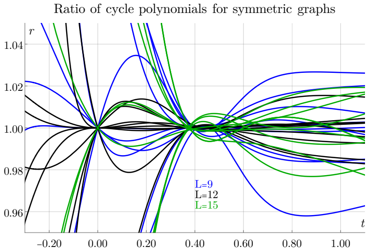

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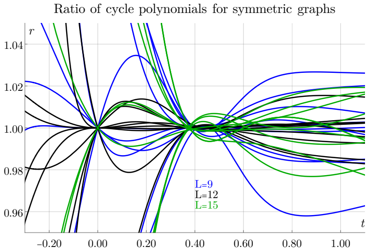
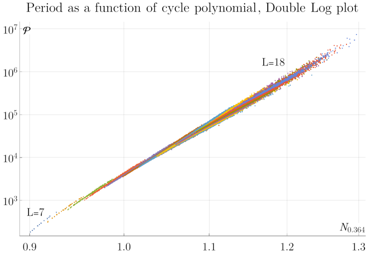
- None of these are invariant under period symmetries, but $N_{0.364}$ almost.
- $\ln \mathcal{P}$ is approximately *the same* linear function of $N_{0.364}$ for all loop orders!

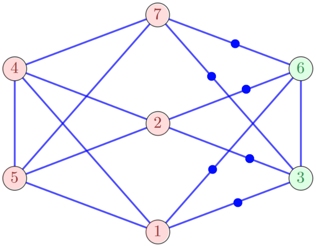
$$\ln \bar{\mathcal{P}}(G) = 28.21 \ln N_{0.364}(G) + 8.258.$$





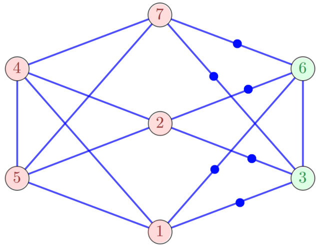
Cuts

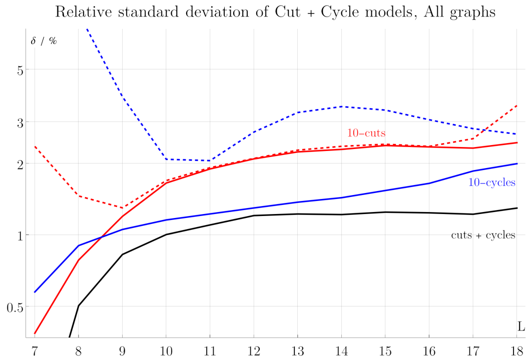
- A (vertex-induced) cut $C \subseteq E_G$ separates the graph into exactly 2 connected components. $c_j \ldots$ number of j-edge cuts. Consider multi-linear model of $\ln(c_j)$.



Cuts

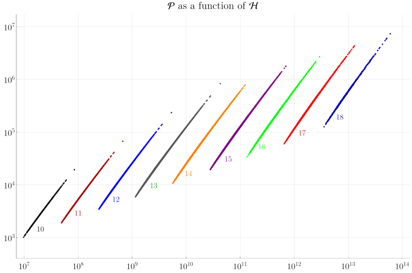
- A (vertex-induced) cut $C \subseteq E_G$ separates the graph into exactly 2 connected components. $c_j \ldots$ number of j-edge cuts. Consider multi-linear model of $\ln(c_j)$.
- Can combine cuts and cycles. Reach $\delta \approx 1.2\%$ for all loop orders.

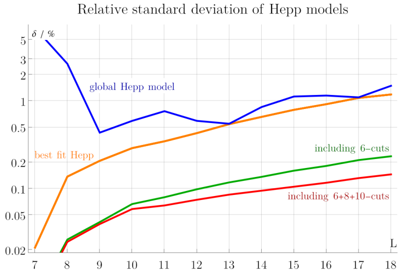




Hepp bound

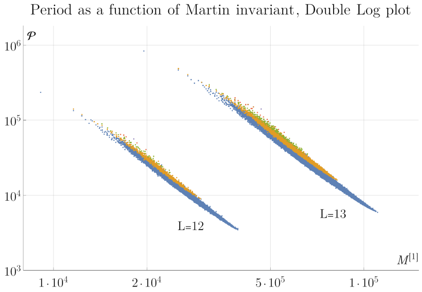
- *Hepp bound* $\mathcal{H}(G)$ [Hepp 1966; Panzer 2022] arises from "tropicalization" of period integral.
- Strongly correlated with period. Low order polynomial function, combined with edge-cuts $\ln(c_j)$, gives $\delta \approx 0.2\%$.
- Computation requires iteration over *all* subgraphs (and/or caching).

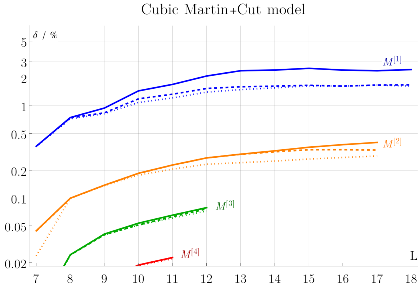




Martin invariant

- *Martin invariant $M^{[k]}$* [Panzer and Yeats 2023] is derivative of $O(N)$ symmetry factor (circuit partition polynomial $J(G,N)$) at $N=-2$ for a graph where every edge is replaced by k parallel edges.
- Linear function of $\ln M^{[1]}$ gives $\delta \approx 4\%$, higher $M^{[k]}$ are much better. k^{th}-order polynomial of $M^{[k]}$ can get very accurate when cobined with cuts $\ln(c_j)$, reach $\delta \ll 0.1\%$.
- Like Hepp, requires recurrence over decompositions and caching.





Machine learning models

- So far: Hand-picked, physically inspired quantities.
- Linear regression of *all* quantities simultaneously gives $\delta \approx 0.1\%$. Quadratic even better.
  But: High computational cost.

Work by **Kimia Shaban**

Machine learning models

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- So far: Hand-picked, physically inspired quantities.
- Linear regression of *all* quantities simultaneously gives $\delta \approx 0.1\%$. Quadratic even better.
  But: High computational cost.
- Trained artificial neural networks (graph convolutional network, graphSAGE) using *just* the graph as input.
- Not as accurate/reproducible as correlation models so far, but very fast once trained.
  Further work in progress.
- Machine learning challenge: The graph *completely* determines the period (perfect accuracy is possible for a clever enough model).
  Training data set freely available, see paulbalduf.com/research.

Summary of correlations and predictions

- The period is correlated with a lot of quantities.
- Some of these correlations allow to predict the period to 1% accuracy within milliseconds.
- Some of them have nice intuitive interpretations (e.g. electrical properties, average distance, etc), but there is hardly any rigorous theory *why* these things are correlated.
- Predicting periods from graphs alone is a challenge for machine learning models.

Now we know a lot of things about periods. Can we actually do something useful with this knowledge?

How can we use our new knowledge?

- The sum of all periods is the (primitive) beta function, which tells us something about how a theory behaves under change of scale.

$$\beta_L^{\text{prim}}(N) := 2 \sum_{\substack{\text{completion } G \\ L \text{ loops}}} \frac{4!(L+2) \cdot J(G,N) \cdot \mathcal{P}(G)}{3^{L+2} N(N+2) |\text{Aut}(G)|}.$$

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- Hard to compute. E.g. at $L=14$, there are $\sim 10^8$ periods. It takes months to compute them all.
- Take a small n-element sample? Standard deviation $\delta(\mathcal{P}) \approx 100\%$ means sampling accuracy

$$\frac{\Delta_{\text{sampling}}\mathcal{P}}{\langle \mathcal{P} \rangle} = \frac{1}{\sqrt{n}}\delta(\mathcal{P}) \approx \frac{1}{\sqrt{n}}.$$

For 3 significant digits ($\Delta_{\text{samp}} < 0.1\%$) we need $n \approx 10^6$.

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  For 3 significant digits ($\Delta_{\text{samp}} < 0.1\%$) we need $n \approx 10^6$.
- $\Rightarrow$ exploit the correlations for a weighted sampling algorithm!

Importance sampling for periods

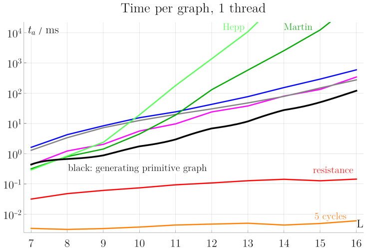
- Idea of importance sampling: If we know a function $\bar{\mathcal{P}}$ which approximates the period *and* $\bar{\mathcal{P}}$ is fast to compute, then:
  1. Evaluate $\langle \bar{\mathcal{P}} \rangle$ in a large sample of size $N_s \cdot n$.
  2. Generate a smaller random sample S of n graphs weighted proportional to $\bar{\mathcal{P}}$. Evaluate $\left\langle \frac{\mathcal{P}}{\bar{\mathcal{P}}} \right\rangle_S$ in this sample.
  3. Law of conditional probability:

$$\langle \mathcal{P} \rangle = \underbrace{\left\langle \frac{\mathcal{P}}{\bar{\mathcal{P}}} \right\rangle_S}_{\substack{\text{slow individually,}\\ \text{but small sample}}} \cdot \underbrace{\langle \bar{\mathcal{P}} \rangle}_{\substack{\text{large sample,}\\ \text{but fast individually}}} .$$

- First factor sampling accuracy is limited by $\delta := \sigma\left(\frac{\mathcal{P}}{\bar{\mathcal{P}}}\right)$ (i.e. accuracy of the prediction function).
- Second factor sampling accuracy is limited by feasible N_s, i.e. by speed of the prediction function $\bar{P}$. Scales like $\frac{1}{\sqrt{N_s}} \propto \sqrt{t_a}$, where $t_a \ldots$ approximation time for one graph.

Speed of the predictions

- Implemented everything in C++, but often naive algorithms.
- Time t_a for approximating one graph depends on model and loop order.
- Resistance and 5-cycles much faster than generating graphs, $\sim 10\mu$s per graph.
- Cuts and cycles similar to generating graphs, $\sim$ 10ms per graph.
- Hepp and Martin much slower, grow faster than all other models, require caching.



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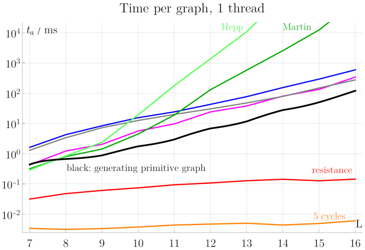
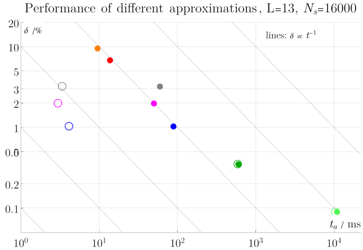
- Hepp and Martin much slower, grow faster than all other models, require caching.
- Most models reach $\delta \propto t_a^{-1}$. Total accuracy scales like $\delta + \sqrt{t_a}$
  $\Rightarrow$ more accurate models are "more than worth it". Use cut + cycle model.





Weighted sampling implementation details

- Generate random graphs, check for primitivity, compute cuts + cycles prediction $\bar{\mathcal{P}}$.
- Metropolis-Hastings sampling algorithm [Hastings 1970; Metropolis et al. 1953]: Have graph G_1. Generate $x \in [0,1]$ uniformly. If $\frac{\bar{\mathcal{P}}(G_2)}{\bar{\mathcal{P}}(G_1)} > x$, accept G_2, else keep G_1.
- Integrate only one in N_s graphs. Maintain a queue of graphs to be integrated while simultaneously generating and weighting new ones.

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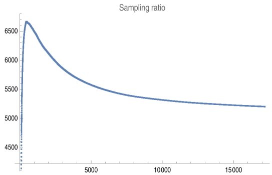
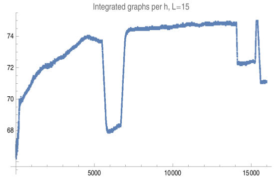
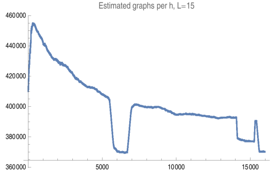
- Integrate only one in N_s graphs. Maintain a queue of graphs to be integrated while simultaneously generating and weighting new ones.
- The program distributes a fixed number of threads threads to the various tasks dynamically, depending on available memory. Chooses sampling factor N_s appropriately. Typically $N_s \approx 5000$ approximations per integration was ideal to balance accuracy of sampling $\left\langle \frac{\mathcal{P}}{\bar{\mathcal{P}}} \right\rangle$ vs $\langle \bar{\mathcal{P}} \rangle$ sampling.

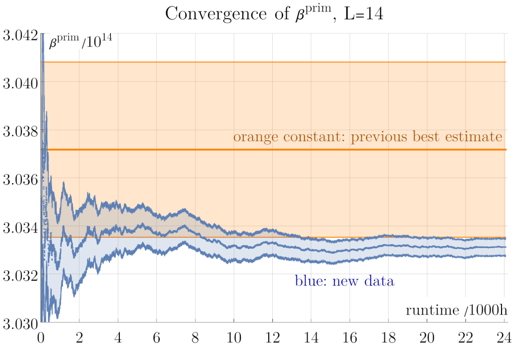






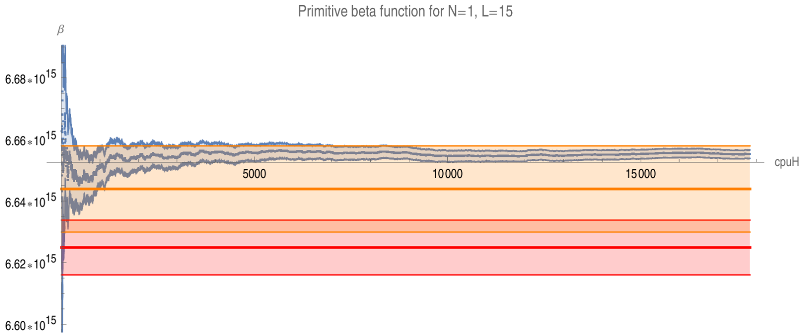
Example results: Primitive beta function for $L = 14$

- Reached 120ppm standard deviation after 24k CPU core h ($<$ 2 weeks walltime).
- Previous work with uniform random sampling took 400k CPU core h for 1063ppm.



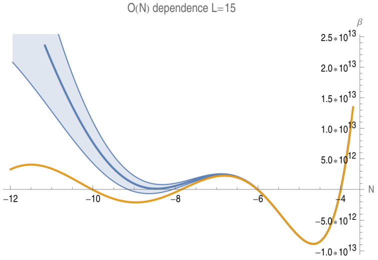
Primitive beta function for $L = 15$

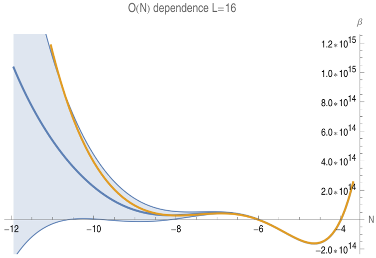
- Reached 200ppm standard deviation after 18k CPU core h $\approx$ 10 days walltime. Previous work with uniform sampling: 1341ppm after 214k CPU core h.
- $\Rightarrow$ Weighted sampling is $\approx 1000\times$ faster than uniform random sampling, or reaches $\approx 35\times$ the accuracy at the same runtime.



$O(N)$ dependence

- Using our results for $\mathcal{P}$, we can also compute the beta function of the $O(N)$-symmetric theory.
- Leading order behavior of $J(G, N)$, and leading graphs for given L, can be identified [B, Thürigen in progress].
- Derivative at $N = -2$ is $M^{[1]}$. Appears to be oscillating with zeros close to $N \in \{-4, -6, -8, \dots\}$.





Conclusion

- We have considered a model quantum field theory, ϕ^4-theory in 4 dimensions, and within a particular class of Feynman graphs, the primitives. The Feynman integral of a primitive graph is a period.
- The number of graphs grows factorially with loop order. The period enjoys symmetries, but they reduce the count moderately, but have interesting combinatorics and invariants.
- Random graphs $\mathcal{G}_{n,4}$ are a fairly good model for large primitive graphs.
- $O(N)$-symmetry involves the circuit partition polynomial, has interesting properties (e.g. Martin invariant, zeros).
- Periods follow an "asymptotic" distribution which has a smooth central part but large outliers.
- The value of the period is correlated with various properties of the graph. Some correlations are physically interesting (e.g. resistance). Often, $\ln(\mathcal{P})$ is a multi-linear function. Accuracy of $\delta \sim 1\%$ can typically be reached.
- The approximations can be used for a weighted sampling algorithm. This is $1000\times$ faster than uniform sampling.

Don't forget to join Mastodon! @paulbalduf@**math**stodon.xyz

References I

Bender, Edward A and E. Rodney Canfield (1978). "The Asymptotic Number of Labeled Graphs with given Degree Sequences". In: *Journal of Combinatorial Theory, Series A* 24.3, pp. 296–307. DOI: 10.1016/0097-3165(78)90059-6.

Bollobás, Béla (1980). "A Probabilistic Proof of an Asymptotic Formula for the Number of Labelled Regular Graphs". In: *European Journal of Combinatorics* 1.4, pp. 311–316. DOI: 10.1016/S0195-6698(80)80030-8.

— (1982). "The Asymptotic Number of Unlabelled Regular Graphs". In: *Journal of the London Mathematical Society* s2-26.2, pp. 201–206. DOI: 10.1112/jlms/s2-26.2.201.

Borinsky, Michael (2017). "Renormalized Asymptotic Enumeration of Feynman Diagrams". In: *Annals of Physics* 385, pp. 95–135. DOI: 10.1016/j.aop.2017.07.009.

— (2023). "Tropical Monte Carlo Quadrature for Feynman Integrals". In: *Annales de l'Institut Henri Poincaré D* 10.4, pp. 635–685. DOI: 10.4171/AIHPD/158. arXiv: 2008.12310.

Borinsky, Michael, Henrik J. Munch, and Felix Tellander (2023). *Tropical Feynman Integration in the Minkowski Regime*. DOI: 10.48550/arXiv.2302.08955. arXiv: 2302.08955 [hep-ph, physics:hep-th, physics:math-ph].

Broadhurst, D. J. and D. Kreimer (1995). "Knots and Numbers in ϕ^4 Theory to 7 Loops and Beyond". In: *International Journal of Modern Physics C* 06.04, pp. 519–524. DOI: 10.1142/S012918319500037X. arXiv: hep-ph/9504352.

Cvitanović, Predrag, B. Lautrup, and Robert B. Pearson (1978). "Number and Weights of Feynman Diagrams". In: *Physical Review D* 18.6, p. 1939. DOI: 10.1103/PhysRevD.18.1939.

Feynman, R. P. (1948). "Space-Time Approach to Non-Relativistic Quantum Mechanics". In: *Rev. Mod. Phys.* 20.2, pp. 367–387. DOI: 10.1103/RevModPhys.20.367.

— (1949). "Space-Time Approach to Quantum Electrodynamics". In: *Physical Review* 76.6, pp. 769–789. DOI: 10.1103/PhysRev.76.769.

References II

Hastings, W. K. (1970). "Monte Carlo Sampling Methods Using Markov Chains and Their Applications". In: *Biometrika* 57.1, pp. 97–109. DOI: 10.1093/biomet/57.1.97.

Hepp, Klaus (1966). "Proof of the Bogoliubov-Parasiuk Theorem on Renormalization". In: *Communications in Mathematical Physics* 2.1, pp. 301–326. DOI: 10.1007/BF01773358.

Hu, Simone et al. (2022). "Further Investigations into the Graph Theory of ϕ^4-Periods and the c_2 Invariant". In: *Annales de l'Institut Henri Poincaré D* 9.3, pp. 473–524. DOI: 10.4171/AIHPD/123.

Kreimer, Dirk (1998). "On the Hopf Algebra Structure of Perturbative Quantum Field Theories". In: *Adv. Theor. Math. Phys.* 2, pp. 303–334. DOI: 10.4310/ATMP.1998.v2.n2.a4. arXiv: q-alg/9707029 [q-alg].

McKane, A. J., D. J. Wallace, and O. F. de Alcantara Bonfim (1984). "Non-Perturbative Renormalisation Using Dimensional Regularisation: Applications to the ε Expansion". In: *Journal of Physics A: Mathematical and General* 17.9, pp. 1861–1876. DOI: 10.1088/0305-4470/17/9/021.

McKane, Alan J. (2019). "Perturbation Expansions at Large Order: Results for Scalar Field Theories Revisited". In: *Journal of Physics A: Mathematical and Theoretical* 52.5, p. 055401. DOI: 10.1088/1751-8121/aaf768. arXiv: 1807.00656 [hep-th].

McKay, B. D. and N. C. Wormald (1984). "Automorphisms of Random Graphs with Specified Vertices". In: *Combinatorica* 4.4, pp. 325–338. DOI: 10.1007/BF02579144.

McKay, Brendan D., Nicholas C. Wormald, and Beata Wysocka (2004). "Short Cycles in Random Regular Graphs". In: *The Electronic Journal of Combinatorics* 11.1, R66. DOI: 10.37236/1819.

Metropolis, Nicholas et al. (1953). "Equation of State Calculations by Fast Computing Machines". In: *The Journal of Chemical Physics* 21.6, pp. 1087–1092. DOI: 10.1063/1.1699114.

Panzer, Erik (2022). "Hepp's Bound for Feynman Graphs and Matroids". In: *Annales de l'Institut Henri Poincaré D* 10.1, pp. 31–119. DOI: 10.4171/aihpd/126.

Panzer, Erik and Karen Yeats (2023). *Feynman Symmetries of the Martin and c_2 Invariants of Regular Graphs*. DOI: 10.48550/arXiv.2304.05299. arXiv: 2304.05299 [hep-th, physics:math-ph].

References III

Schnetz, Oliver (2010). "Quantum Periods: A Census of ϕ^4-Transcendentals". In: *Commun.Num.Theor.Phys.* 4, pp. 1–48. arXiv: 0801.2856.

Symanzik, Kurt (1970). "Small Distance Behaviour in Field Theory and Power Counting". In: *Communications in Mathematical Physics* 18.3, pp. 227–246. DOI: 10.1007/BF01649434.

Wormald, Nicholas C. (1981). "The Asymptotic Connectivity of Labelled Regular Graphs". In: *Journal of Combinatorial Theory, Series B* 31.2, pp. 156–167. DOI: 10.1016/S0095-8956(81)80021-4.