UNIVERSITY OF OXFORD
Mathematical Institute

Combinatorics, Resurgence and Algebraic Geometry in Quantum Field Theory
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Variations of single-kernel Dyson-Schwinger equations

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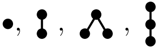
Slides, links, etc. available from paulbalduf.com/research

Connes-Kreimer Hopf algebra

- Hopf algebra $\mathcal{H}$ spanned by forests of rooted trees, i.e. generated by rooted trees  $, \ldots$ over $\mathbb{Q}$

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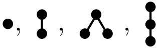
- Hopf algebra $\mathcal{H}$ spanned by forests of rooted trees, i.e. generated by rooted trees , . . . over $\mathbb{Q}$, where
  - Multiplication $m : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ is disjoint union,
  - Unit $\mathbb{1} : \mathbb{Q} \to \mathcal{H}$, where the empty tree is identified with $\mathbb{1} \in \mathcal{H}$,
  - Counit $\tilde{\mathbb{1}} : \mathcal{H} \to \mathbb{Q}$ is 0 except for $\mathcal{H} \ni q\mathbb{1} \mapsto q \in \mathbb{Q}$,
  - Coproduct is sum over *admissible cuts* c that separate the root R_c from the pruned part P_c,

$$\Delta[h] = h \otimes \mathbb{1} + \mathbb{1} \otimes h + \sum_{\emptyset \neq c \in C} P^c[h] \otimes R^c[h].$$

  - Notation: Action on Hopf algebra $[h]$ for $h \in \mathcal{H}$, ordinary argument (x).
  - Graded by number of vertices, connected $\Rightarrow$ antipode S follows recursively from Δ by $\Delta S[h] = (S \otimes S)\,\text{flip}\,\Delta[h]$ and $S[\mathbb{1}] = 0$. Concretely

$$S[h] = -h - \sum_{\emptyset \neq c \in C} S\big(P^c[h]\big) R^c[h].$$

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  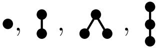 , . . . over $\mathbb{Q}$, where

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- Distinct drawings of the same tree are equivalent (i.e. not plane trees).
- In real physics, vertices are decorated, but not in this talk.

Connes-Kreimer Hopf algebra, example

- Let $\Delta[h] = h \otimes \mathbb{1} + \mathbb{1} \otimes h + \tilde{\Delta}[h]$.

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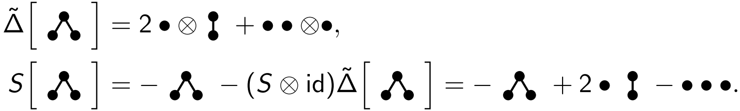

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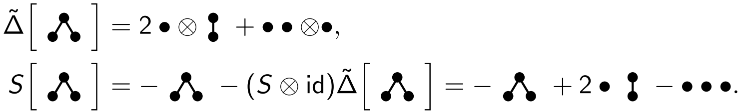

- Multiplicativity $\Delta[h_1 h_2] = \Delta[h_1]\Delta[h_2]$ implies

$$\Delta[\underbrace{\bullet\bullet\cdots\bullet}_{n \text{ factors}}] = \sum_{j=0}^{n} \binom{n}{j} \underbrace{\bullet\bullet\cdots\bullet}_{n-j \text{ factors}} \otimes \underbrace{\bullet\bullet\cdots\bullet}_{j \text{ factors}}.$$

Hochschild cycle

- Operator $B_+ : \mathcal{H} \to \mathcal{H}$ "adds a new root". E.g.

$$B_+[\mathbb{1}] = \bullet, \qquad B_+[\bullet\bullet\bullet] =$$  $$, \qquad B_+\Big[$$  $$\Big] =$$  .

- B_+ is a Hochschild 1-cocycle,

$$\Delta B_+[h] = B_+[h] \otimes \mathbb{1} + (\mathrm{id} \otimes B_+)\Delta[h].$$

- The Connes-Kreimer Hopf algebra $\mathcal{H}$ together with B_+ is universal among Hopf algebras with a 1-cocycle:

Theorem (Universal property [Connes and Kreimer 1999])

If A is a connected graded Hopf algebra, and Λ a Hochschild 1-cocycle on A, then there is a unique bialgebra morphism $\phi : \mathcal{H} \to A$ such that $\phi B_+ = \Lambda \phi$.

Rooted trees and Feynman graphs

- For our physical application, a rooted tree is a symbol for a certain Feynman graph.
- We start from some primitive "kernel" graph Γ, symbolized by $\bullet$, then "inserting subtrees" below $\bullet$ amounts to inserting subgraphs into Γ.

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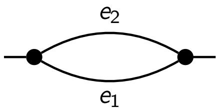
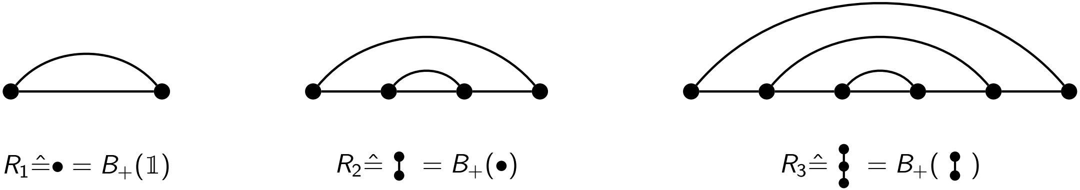
- In this talk, mostly consider the kernel

$\Gamma =$ 

- This graph is a quantum correction to a propagator (has 2 external legs). Can insert it into one of the internal edges e_1, e_2. For now, insert into e_1 only.

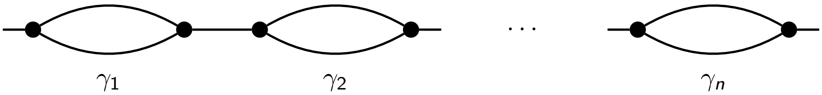
Example: Ladder trees / Rainbows

- Operator $B_+(\gamma)$ inserts γ into the lower edge of Γ.
- For every order n (=number of vertices), there is 1 ladder tree  $\}n$, and there is 1 rainbow Feynman graph R_n.



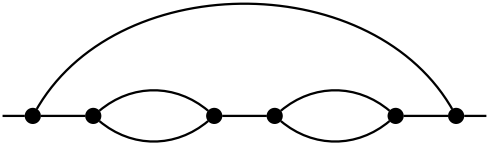
Example: Corollas / Chains

- Product $m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ means "join at external leg" (in this simple case).
- Product $\underbrace{\bullet \bullet \cdots \bullet}_{n \text{ factors}}$ maps to chain of multiedge graphs C_n.

$C_n =$ 

Example: A more complicated graph

- C_2 is the chain of two multiedges. Corresponds to $\bullet\bullet \in \mathcal{H}$.
- $B_+[\bullet\bullet] =$  corresponds to the shown graph, where C_2 is inserted into the lower edge of the kernel.



Characters

- A *character* ϕ is a morphism $\mathcal{H} \to A$ into some algebra A, i.e.

$$\phi\Big[\underbrace{h_1 \cdot h_2}_{\text{multiplication in } \mathcal{H}}\Big] = \underbrace{\phi[h_1] \cdot \phi[h_2]}_{\text{multiplication in } A}$$

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- Characters are a group, with operation "convolution product"

$$(\phi_1 \star \phi_2)[h] := m_A(\phi_1 \otimes \phi_2)\Delta[h].$$

- Inverse: $\phi^{-1} := \phi S$, then $\phi^{-1} \star \phi = \mathbb{1}\tilde{\mathbb{1}}$.

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- Inverse: $\phi^{-1} := \phi S$, then $\phi^{-1} \star \phi = \mathbb{1}\tilde{\mathbb{1}}$.
- σ is an *infinitesimal character* if $\sigma[h_1 \cdot h_2] = \sigma(h_1) \cdot \tilde{\mathbb{1}}[h_2] + \sigma(h_2) \cdot \tilde{\mathbb{1}}[h_1]$.
- If σ is an infinitesimal character, then $\exp_\star(\sigma) := \tilde{\mathbb{1}} + \sigma + \frac{1}{2}\sigma \star \sigma + \ldots$ is a character. All characters are of this form (e.g. [Cartier and Patras 2021]).

Feynman rules

- *Feynman rules* $\mathcal{F}_\mathcal{R}$ are a character. They map to power series (=the binomial coalgebra) in variable $L = \ln \frac{p^2}{s_0}$. Notation $\mathcal{F}_\mathcal{R}[X](L)$ for $X \in \mathcal{H}$.
- Being a character is a physical axiom: (Probability) amplitude of independent processes should be product, $\mathcal{F}_\mathcal{R}[X_1 X_2] = \mathcal{F}_\mathcal{R}[X_1]\mathcal{F}_\mathcal{R}[X_2]$

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- These are *renormalized* Feynman rules, $\mathcal{F}_\mathcal{R}[X] = (\mathcal{R} \circ \mathcal{F} \circ S \star \mathcal{F})[X]$ (more on that later).
- Infinitesimal Feynman rules σ, related via

$$\mathcal{F}_\mathcal{R}[X](L) = \exp^\star(L\sigma)[X] = \tilde{\mathbb{1}}[X] + \sigma[X]L + \frac{1}{2}(\sigma \star \sigma)[X]L^2 + \dots$$

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- Under Feynman rules, the cocycle B_+ maps to an integral operator, the Feynman integral. schematically

$$\mathcal{F}_\mathcal{R}\Big[B_+[\gamma]\Big](L) = (1 - \mathcal{R}) \int \mathrm{d}k\ K(L,k)\mathcal{F}_\mathcal{R}[\gamma](k).$$

Function $K(L,k)$ is the integrand of the kernel graph Γ.

Analytical Dyson-Schwinger equations

- Fundamental quantum principle: All processes that *can* take place, *will* take place (simultaneously).
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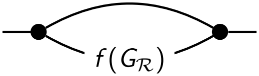
- Integral equation for the renormalized Green function $G_{\mathcal{R}}(\alpha, L)$ as a power series in expansion parameter α. Called *analytical DSE*.
- We leave out all (infinitely many) but the very first one kernel graph

$$G_{\mathcal{R}} = 1 + \alpha(1 - \mathcal{R})$$



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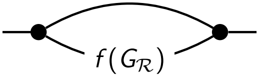

  f denotes some function to be discussed in the following.
- On the level of rooted trees, "inserting into itself" means that we have a fixed-point equation for some object $X(\alpha) \in \mathcal{H}$ under the insertion operator B_+. Called *combinatorial DSE*, where $\mathcal{F}_\mathcal{R}[X(\alpha)](L) =: G_\mathcal{R}(\alpha, L)$.

Combinatorial Dyson-Schwinger equation

- Recall $B_+[\gamma]$ means "attach γ below a new root".
- Let $X(\alpha)$ be a formal power series (parameter α) with coefficients in $\mathcal{H}$. Expected structure:

$$X(\alpha) = \mathbb{1} + \alpha B_+\Big[f[X]\Big].$$

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  Which function f can we take?
- Recall renormalized Feynman rules $\mathcal{F}_{\mathcal{R}}[X] = (\mathcal{R}\mathcal{F}S \star \mathcal{F})[X]$, i.e. left side of coproduct determines counterterm.
- Physically: Interpret counterterm as reparametrization, must consist of "the same" series as X itself ("multiplicative renormalizability").
- More precisely: If $X(\alpha) = \sum_n \alpha^n x_n$, the x_j should generate a sub Hopf algebra (i.e. be closed under coproduct).

Sub Hopf algebras from combinatorial DSEs

Theorem ([Foissy 2008])

Let $X(\alpha) = \alpha x_1 + \alpha^2 x_2 + \dots$. *Then, the Dyson-Schwinger equation* $X = \mathbb{1} + \alpha B_+\left[f[X]\right]$ *generates a sub Hopf algebra in exactly two cases:*

1. $X = \mathbb{1} + \alpha B_+\left[X^{1+w}\right]$ *for* $w \in \mathbb{R}$, *and then*

$$\Delta X(\alpha) = \sum_{n=0}^{\infty} X^{wn+1}(\alpha) \otimes \alpha^n x_n,$$

2. *or* $X = \mathbb{1} + \alpha B_+\left[e^{X-1}\right]$, *in which case*

$$\Delta X(\alpha) = X(\alpha) \otimes \mathbb{1} + \sum_{n=1}^{\infty} e^{(X-1)n} \otimes \alpha^n x_n.$$

All physically known DSEs fall into the first case, with w small positive or negative integers (or maybe half-integers).

The exponential DSE, order 0

$$X(\alpha) = \mathbb{1} + \alpha B_+\left(e^{X(\alpha)-1}\right)$$

- Series solution starts with

$$X = 1 + \dots$$

- Exponential:

$$e^{X-1} = e^{0+\dots} = 1 + \dots$$

- Coproduct:

$$\Delta X = \mathbb{1} \otimes \mathbb{1} + \dots$$

The exponential DSE, order 1

$$X(\alpha) = \mathbb{1} + \alpha B_+\left(e^{X(\alpha)-1}\right)$$

- Series solution starts with

$$X = 1 + \bullet\alpha + \dots$$

- Exponential:

$$e^{X-1} = e^{\alpha\bullet+\dots} = 1 + \bullet\alpha + \dots$$

- Coproduct:

$$\Delta X = \mathbb{1}\otimes\mathbb{1} + \bullet\alpha\otimes\mathbb{1} + \alpha\mathbb{1}\otimes\bullet + \dots$$

The exponential DSE, order 2

$$X(\alpha) = \mathbb{1} + \alpha B_+\left(e^{X(\alpha)-1}\right)$$

- Series solution starts with

$$X = 1 + \bullet\alpha + \begin{smallmatrix}\bullet\\ \vert\\ \bullet\end{smallmatrix}\,\alpha^2 + \dots$$

- Exponential:

$$e^{X-1} = 1 + \bullet\alpha + \left(\begin{smallmatrix}\bullet\\ \vert\\ \bullet\end{smallmatrix} + \frac{1}{2}\bullet\bullet\right)\alpha^2 + \dots$$

- Coproduct:

$$\Delta X = X \otimes \mathbb{1} + \alpha(\mathbb{1} + \alpha\bullet + \dots) \otimes \bullet$$
$$+ \alpha^2(\mathbb{1} + \dots) \otimes \begin{smallmatrix}\bullet\\ \vert\\ \bullet\end{smallmatrix} + \dots$$

The exponential DSE, Order 4

$$X(\alpha) = \mathbb{1} + \alpha B_+\left(e^{X(\alpha)-1}\right)$$

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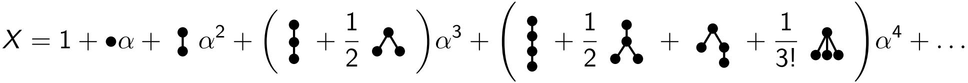

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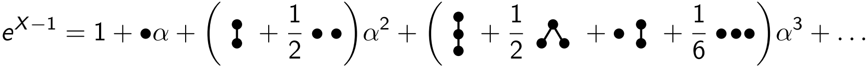
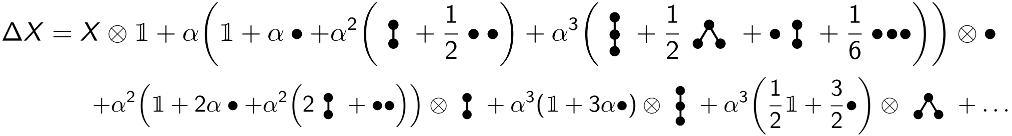


- Coproduct:



$$= X \otimes \mathbb{1} + e^{X-1} \otimes \alpha x_1 + e^{2(X-1)} \otimes \alpha^2 x_2 + e^{3(X-1)} \otimes \alpha^3 x_3 + \dots$$

Polynomial DSEs

$$X(\alpha) = \mathbb{1} + \alpha B_+ \left[X^{w+1}(\alpha) \right]$$

- $w = 0$ (*linear DSE*) leads to ladder trees (rainbow Feynman graphs)

$$X(\alpha) = \mathbb{1} + \alpha \bullet + \alpha^2 \begin{smallmatrix}\bullet\\ | \\ \bullet\end{smallmatrix} + \alpha^3 \begin{smallmatrix}\bullet\\ | \\ \bullet\\ | \\ \bullet\end{smallmatrix} + \ldots = \sum_{n=0}^{\infty} \alpha^n \left. \begin{smallmatrix}\bullet\\ | \\ \bullet\\ \vdots \\ \bullet\end{smallmatrix} \right\} n$$

- $w = -1$ is a trivial-non-recursive DSE: $X(\alpha) = \mathbb{1} + \alpha B_+(\mathbb{1}) = \mathbb{1} + \alpha \bullet$. Physically, this gives just the kernel graph $\bullet$, without inserting anything
- $w = -2$ means inserting $\frac{1}{X(\alpha)}$. This is for propagator corrections (i.e. inserting geometric series). For $w = -2$, $X(\alpha)$ is the sum of *all* rooted trees.
- Any w that is not an integer ≥ -1 leads to a sum of *all* rooted trees with some weighting.

Renormalization group equation for polynomial DSE

- Consider the Feynman rules at a shifted energy scale $L+\delta$,

$$\mathcal{F}_{\mathcal{R}}[X](\delta+L) = e^{\star\sigma(\delta+L)}[X] = \left(e^{\star\sigma\delta} \star e^{\star\sigma L}\right)[X] = m\Big(\mathcal{F}_{\mathcal{R}}(\delta) \otimes \mathcal{F}_{\mathcal{R}}(L)\Big)\Delta[X]$$

- Compute derivative w.r.t. δ at the point $\delta = 0$, obtain differential equation.
- Use known coproduct $\Delta[X]$ for DSE solutions.

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- Encounter *running coupling* $\tilde{\alpha}(L) = \alpha G^w_\mathcal{R}(\alpha, L)$. Introduce *renormalization group functions*

$$\gamma(\alpha) := \partial_L G_\mathcal{R}(\alpha, L)\Big|_{L=0}, \qquad \beta(\alpha) = \partial_L \tilde{\alpha}(\alpha, L)\Big|_{L=0} = w\alpha\gamma(\alpha).$$

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- Find *Callan-Symanzik equation* [Callan 1970; Symanzik 1970], define $\mathcal{D}=(1+w\alpha\partial_\alpha)$

$$\partial_L G_\mathcal{R}(\alpha,L) = (\gamma(\alpha)+\beta(\alpha)\cdot\partial_\alpha)G_\mathcal{R}(\alpha,L) = \gamma(\alpha)\mathcal{D}G_\mathcal{R}(\alpha,L).$$

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- Find *Callan-Symanzik equation* [Callan 1970; Symanzik 1970], define $\mathcal{D}=(1+w\alpha\partial_\alpha)$

$$\partial_L G_\mathcal{R}(\alpha,L) = (\gamma(\alpha)+\beta(\alpha)\cdot\partial_\alpha)G_\mathcal{R}(\alpha,L) = \gamma(\alpha)\mathcal{D}G_\mathcal{R}(\alpha,L).$$

- Expand $G_\mathcal{R}(\alpha,L) =: 1+\sum_j\gamma_j(\alpha)L^j$, then

$$\gamma_1(\alpha)=\gamma(\alpha), \qquad \gamma_k(\alpha) = \frac{1}{k}\gamma(\alpha)\mathcal{D}\gamma_{k-1}(\alpha).$$

Details e.g. [Bergbauer and Kreimer 2006].

Exponential DSE vs polynomial DSE

- DSE

$$X = \mathbb{1} + \alpha B_+(X^{w+1})$$

- running coupling

$$\tilde{\alpha} = \alpha G_{\mathcal{R}}^{w}, \quad \beta(\alpha) = w\alpha\gamma(\alpha)$$

- Callan-Symanzik equation

$$\partial_L G_{\mathcal{R}}(\alpha, L) = (\gamma(\alpha) + \beta(\alpha)\cdot\partial_\alpha) G_{\mathcal{R}}(\alpha, L)$$

- Expansion functions, $\mathcal{D} = (1 + w\alpha\partial_\alpha)$

$$\gamma_1 = \gamma, \quad \gamma_k = \frac{1}{k}\gamma\mathcal{D}\gamma_{k-1}$$

- DSE

$$X = \mathbb{1} + \alpha B_+\left(e^{X-1}\right)$$

- running coupling

$$\tilde{\alpha} = \alpha e^{G_{\mathcal{R}}-1}, \quad \beta(\alpha) = \alpha\gamma(\alpha)$$

- "Callan-Symanzik equation"

$$\partial_L G_{\mathcal{R}}(\alpha, L) = \gamma(\alpha) + \beta(\alpha)\partial_\alpha G_{\mathcal{R}}(\alpha, L).$$

(May consider $Y := e^{G_{\mathcal{R}}} = eQ_{\mathcal{R}}$ [Foissy 2008])

- Expansion functions, $\mathcal{D} = \alpha\partial_\alpha$

$$\gamma_1 = \gamma, \quad \gamma_k = \frac{1}{k}\gamma\mathcal{D}\gamma_{k-1}$$

Summary so far

- A Dyson-Schwinger equation is a mathematical formalization of the "quantum principle". For us, its validity is an axiom.
- Consistent multiplicative renormalization requires that the solution of the DSE generates a sub Hopf algebra. Another axiom.
- This leaves two possible forms of DSE

$$X = \mathbb{1} + \alpha B_+\left(X^{w+1}\right), \quad \text{or} \quad X = \mathbb{1} + \alpha B_+\left(e^{X-1}\right).$$

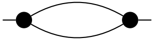
- They lead to slightly different renormalization group equations, operator either $\mathcal{D} = 1 + w\alpha\partial_\alpha$ or $\mathcal{D} = \alpha\partial_\alpha$. In both cases

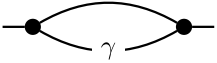
$$\gamma_1 = \gamma, \quad \gamma_k = \frac{1}{k}\gamma\mathcal{D}\gamma_{k-1}$$

for the expansion

$$G_\mathcal{R}(\alpha, L) = 1 + \sum_{n=1}^\infty \gamma_n(\alpha) L^n.$$

B_+ as integral operator

- Developed over 25 years Broadhurst and Kreimer 2000; Broadhurst and Kreimer 2001; Kreimer and Yeats 2006; Kreimer 2008; Yeats 2008; Yeats 2011; Kreimer and Panzer 2013; Balduf 2024.
- Recall B_+ means "insertion of subgraphs". $B_+^\Gamma(\gamma)$ is a Feynman integral of the primitive graph Γ, where a subgraph γ has been inserted.
- E.g. concretely for insertion of γ into 1-loop multiedge $\Gamma =$ 

$\mathcal{F}\left[B_+^\Gamma(\gamma)\right](p^2) =$ 

$= \int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{1}{k^2} \mathcal{F}[\gamma](k^2) \frac{1}{(p+k)^2}.$

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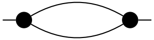
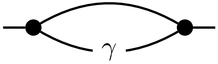
- E.g. concretely for insertion of γ into 1-loop multiedge $\Gamma =$ 

$$\mathcal{F}\big[B_+^\Gamma(\gamma)\big](p^2) =$$

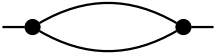


$$= \int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{1}{k^2} \mathcal{F}[\gamma](k^2) \frac{1}{(p+k)^2}.$$

- $\mathcal{F}_\mathcal{R}[\gamma](k^2)$ is proportional to $(k^2)^{-\ell\epsilon}$. Inserting γ is equivalent to changing the power of k^2 in the integral.
- Integral where edges have arbitrary propagator powers, *Mellin transform* $F_\Gamma(\rho) = \sum_k c_k \rho^k$

$$\mathcal{F}\Big[B_+^\Gamma(\mathcal{F}_\mathcal{R}[\gamma])\Big](L) = -\mathcal{F}_\mathcal{R}[\gamma](\partial_\rho) e^{L\rho} F_\Gamma(\rho)\Big|_{\rho=0}.$$

Mellin transform of the 1-loop multiedge



- Mellin transform ≈ Feynman integral $\mathcal{F}[\Gamma]$ in analytic regularization, propagator powers $\nu_e = 1-\rho_e$, evaluated at $p^2 = s_0$. Leave out all inessential prefactors

$$\mathcal{F}[\Gamma](s) \propto \int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{1}{(p^2)^{1-\rho_1}} \frac{1}{((p+k)^2)^{1-\rho_2}}\Big|_{p^2=s_0} \propto \int_0^\infty \mathrm{d}a_1 \int_0^\infty \mathrm{d}a_2 \; a_1^{-\rho_1} a_2^{-\rho_2} \frac{\exp\left(-\frac{a_1 a_2 s}{a_1+a_2}\right)}{(a_1+a_2)^{\frac{D}{2}}}$$

$$= \frac{\Gamma\left(-\rho_1-\rho_2+2-\frac{D}{2}\right)\Gamma\left(\frac{D}{2}-1+\rho_1\right)\Gamma\left(\frac{D}{2}-1+\rho_2\right)}{\Gamma(D-2+\rho_1+\rho_2)\Gamma(1-\rho_1)\Gamma(1-\rho_2)} =: F_{M^{(1)}}(\rho_1,\rho_2).$$

- Insert only into e_1 (set $\rho_2 = 0$). Evaluate at $D=4$ or $D=6$.

$$F_{M^{(1)}}(\rho,0)\Big|_{D=4} = \frac{-1}{\rho(1+\rho)}, \qquad F_{M^{(1)}}(\rho,0)\Big|_{D=6} = \frac{1}{\rho(1+\rho)(2+\rho)(3+\rho)}.$$

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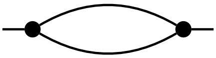
$\Gamma =$ 

- Mellin transform $\approx$ Feynman integral $\mathcal{F}[\Gamma]$ in analytic regularization, propagator powers $\nu_e = 1-\rho_e$, evaluated at $p^2 = s_0$. Leave out all inessential prefactors

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From combinatorial DSE to differential equation in 4 steps

1. Kinematic renormalization (MOM) amounts to subtraction at $L = 0$, therefore

$$\mathcal{F}_{\mathcal{R}}\left[B_+^\Gamma(\gamma)\right](L) = -\mathcal{F}_{\mathcal{R}}[\gamma](\partial_\rho)\left(e^{L\rho} - 1\right)F_\Gamma(\rho)\Big|_{\rho=0}.$$

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2. Use this in DSE $\mathcal{F}_{\mathcal{R}}[X] = 1+\alpha\mathcal{F}_{\mathcal{R}}\left[B_+[X^{1+w}]\right]$

$$G_{\mathcal{R}}(\alpha,L) = 1-\alpha\left(G_{\mathcal{R}}^{1+w}(\alpha,\partial_\rho)e^{L\rho}F(\rho) - G_{\mathcal{R}}^{1+w}(\alpha,\partial_\rho)F(\rho)\right)\Big|_{\rho=0}. \qquad (*)$$

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4. Apply this differential operator to both sides of the DSE $(*)$, use Callan-Symanzik equation $\partial_L G_\mathcal{R} = \gamma\mathcal{D}G_\mathcal{R}$, extract order zero in L

$$\frac{1}{\rho\cdot F(\rho)}\Big|_{\rho\to\gamma\mathcal{D}}\gamma(\alpha) = -\alpha.$$

DSE in MOM as ODE

- ODE for anomalous dimension, where $\mathcal{D} = 1 + w\alpha\partial_\alpha$ or $\mathcal{D} = \alpha\partial_\alpha$ and $\bar{F}(\rho) = \rho F(\rho)$

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- E.g. Multiedge in 4D $F(\rho) = \frac{-1}{\rho(1+\rho)}$ for $w = -2$

$$(1 + \gamma(1 - 2\alpha\partial_\alpha))\gamma = \alpha \qquad \Rightarrow \qquad \gamma(\alpha) = \alpha + \alpha^2 + 4\alpha^3 + 27\alpha^4 + 248\alpha^5 + \dots$$

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- Unless $w = -1$ or $w = 0$, series is divergent.
- Exponential DSE: Series solution of $\gamma = \alpha - \alpha\gamma\gamma'$ is known as [OEIS A088716]

$$\gamma(\alpha) = \alpha - \alpha^2 + 3\alpha^3 - 14\alpha^4 + 85\alpha^5 - 621\alpha^6 + 5236\alpha^7 \mp \dots = \sum_n c_n \alpha^n$$

$$c_n \sim S \cdot (-1)^{n+1}\Gamma(n+1)\big(1 + \mathcal{O}\big(n^{-1}\big)\big).$$

Stokes constant $S \approx 0.21795\dots$.

Transseries approach to asymptotics

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- Have perturbative solution of ODE,

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- Ansatz for non-perturbative part [Borinsky, Dunne, and Meynig 2021; Borinsky and Dunne 2020]

$$\gamma^{\text{non-pert}}(\alpha) = \alpha^{\mu(w)} \exp\left(\frac{\lambda(w)}{\alpha}\right)\left(1 + b^{(1)}(w)\alpha + b^{(2)}(w)\alpha^2 + \dots\right).$$

- Insert $\gamma(\alpha) = \gamma^{\text{pert}}(\alpha) + \gamma^{\text{non-pert}}(\alpha)$ into ODE, for 4D model

$$(1 + \gamma(\alpha)(1 + w\alpha\partial_\alpha))\gamma(\alpha) = \alpha.$$

- Expand in powers of $\gamma^{\text{non-pert}}(\alpha)$. Obtain equations for parameters of $\gamma^{\text{non-pert}}(\alpha)$.

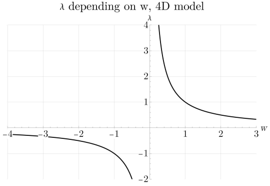
Transseries parameters for 4D model

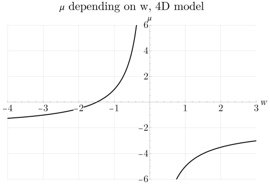
- Insert ansatz

$$\gamma^{\text{non-pert}}(\alpha) = \alpha^{\mu(w)} \exp\left(\frac{\lambda(w)}{\alpha}\right)\left(1 + b^{(1)}(w)\alpha + b^{(2)}(w)\alpha^2 + \ldots\right).$$

- Find parameters as functions of w

$$\lambda(w) = \frac{1}{w}, \quad \mu(w) = -\frac{3+2w}{w}, \quad b^{(1)}(w) = \frac{(1+w)(1+3w)}{w},$$
$$b^{(2)}(w) = \frac{(1+w)(1+5w+3w^2-5w^3)}{2w^2},$$
$$b^{(3)}(w) = \frac{(1+w)(1+5w-4w^2-20w^3+45w^4+81w^5)}{6w^3}, \ldots$$





- Corresponding asymptotic growth:

$$c_n \sim S(w) \cdot \frac{1}{(-\lambda(w))^n} \cdot \Gamma\left(n - \mu(w)\right)\left(1 + \frac{-\lambda(w) \cdot b^{(1)}(w)}{(n-\mu-1)} + \ldots\right)$$

Stokes constant for 4D model

- Stokes constant $S(w)$ in $c_n \sim S(w) \cdot \frac{1}{(-\lambda(w))^n} \cdot \Gamma\Big(n - \mu(w)\Big)(1 + \ldots)$
  Determined numerically from series coefficients [Balduf 2024]

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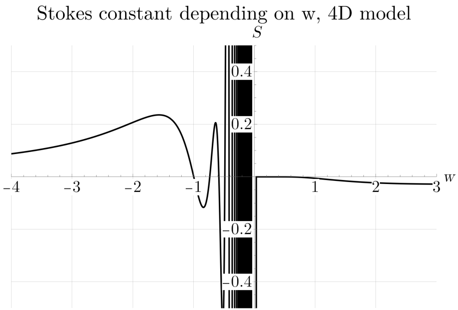

- Note smooth limit $S \to 0$ as $w \to 0^+$ to linear DSE, divergence $w \to 0^-$.

Resummation of 4D model

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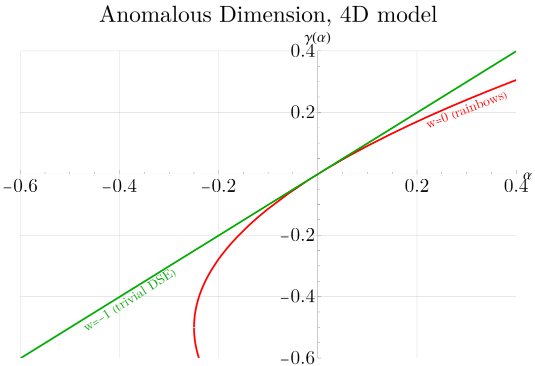
- Perturbative "solution" is asymptotic power series $\Rightarrow$ is not a "solution" in the physical sense, gives no finite prediction.
- ODE is of first order, has 1 free boundary condition

$$(1+\gamma(\alpha)(1+w\alpha\partial_\alpha))\gamma(\alpha)=\alpha.$$

- Note that $\partial_\alpha \gamma = \frac{1}{w\gamma(\alpha)\cdot\alpha}\left(\alpha-\gamma-\gamma^2\right)$ is singular at the origin.
- More details and systematic resummation in [Borinsky and Dunne 2020; Borinsky, Dunne, and Meynig 2021; Borinsky and Broadhurst 2022].

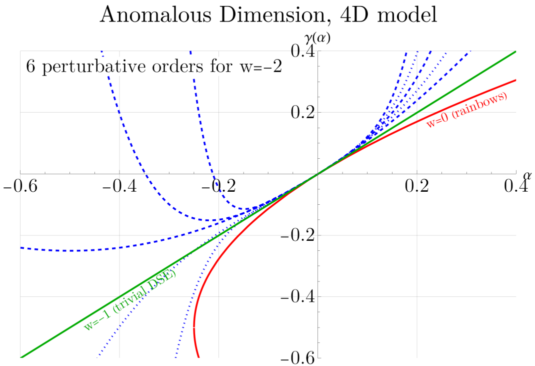
Analytic solutions of 4D model



$$X = 1 + \alpha B_+[X^{w+1}] \qquad \gamma + \gamma(1 + w\alpha\partial_\alpha)\gamma = \alpha$$

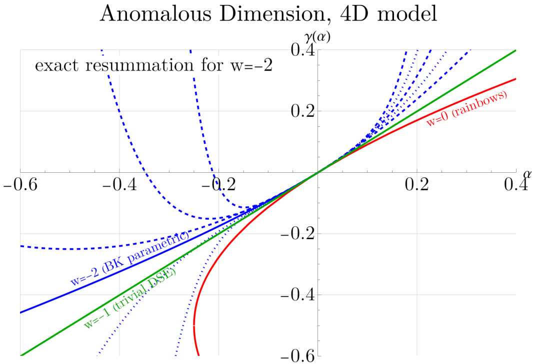
- For $w = -1$, the DSE is not even recursive. Kernel graph is exact solution, $\gamma(\alpha) = \alpha$.
- For $w = 0$, algebraic equation instead of ODE, exact solution $\gamma = \frac{\sqrt{1+4\alpha}-1}{2}$ (red line).

Asymptotic series solution for $w = -2$



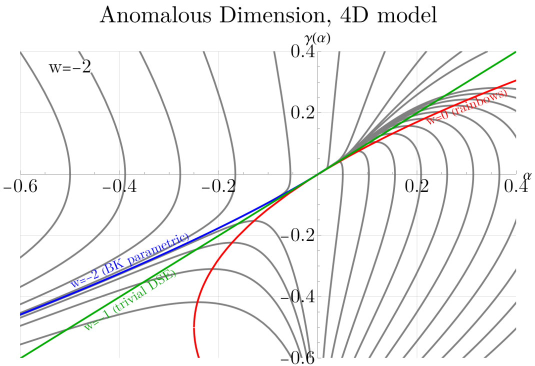
- For $w = -2$, (divergent) perturbative series starts with $\gamma(\alpha) = \alpha + \alpha^2 + 4\alpha^3 + 27\alpha^4 + 248\alpha^5 + 2830\alpha^6$.
- "Physical" domain for this model is $\alpha < 0$ since we used $X = 1 + \alpha B_+\left[X^{1+w}\right]$.

Resummed solution for $w = -2$



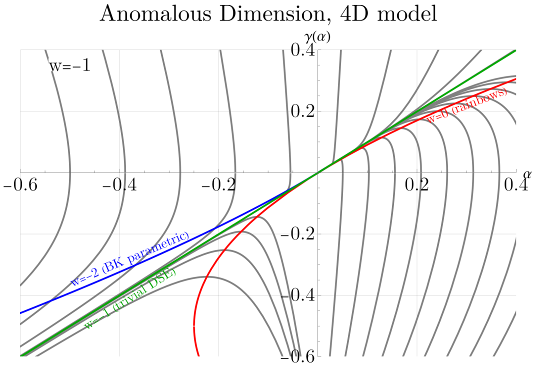
- Resummed exact solution from [Broadhurst and Kreimer 2001].

Numerical solutions for $w = -2$



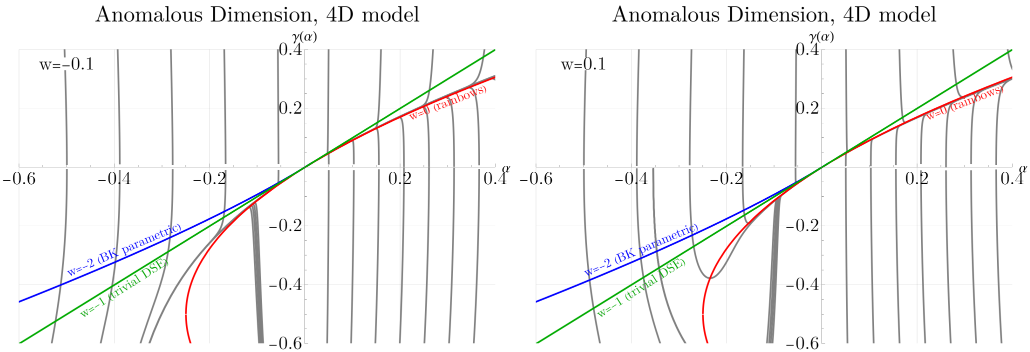
- Notable qualitative difference between $\alpha > 0$ (all solutions exponential small as $\alpha \to 0$) and $\alpha < 0$ (exponentially large).
- Linear DSE solution (red curve) is locus of vanishing derivative of $\gamma(\alpha)$ [Yeats 2008].

Numerical solutions for $w = -1$ (trivial DSE)



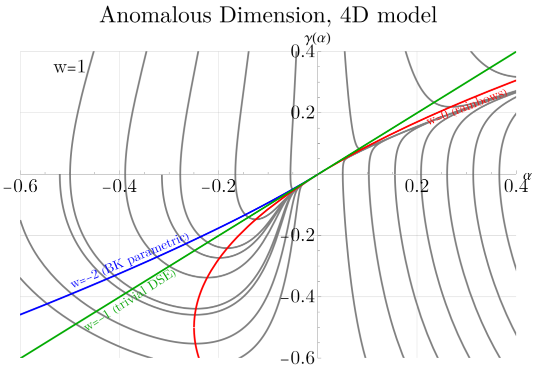
- Physically, the solution at $w = -1$ is unique: $\gamma(\alpha) = \alpha$.
- The ODE has many other solutions. They look qualitatively similar to other w.
- Again, linear solution gives locus of $\partial_\alpha \gamma = 0$.

Numerical solutions close to $w = 0$



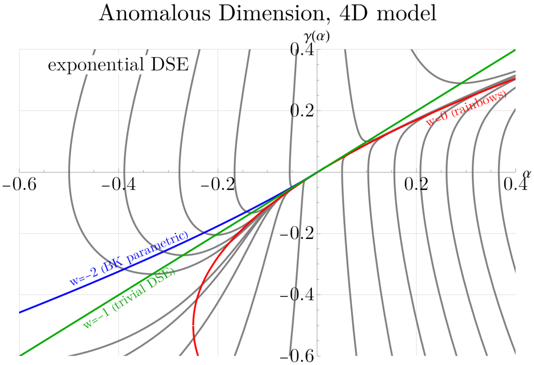
- Recall exotic behavior of Stokes constant near $w = 0$.
- Again, linear solution $w = 0$ gives locus of $\partial_\alpha \gamma = 0$.
- Crossing $w = 0$, the slope "flips" $\Rightarrow$ qualitative change of resurgence functions.

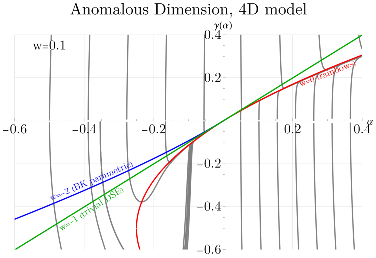
Numerical solutions for $w = +1$



- $w = +1$ would be a vertex DSE, physical at $\alpha > 0$.
- Now, solutions at $\alpha < 0$ are exponentially small for $\alpha \to 0$.

Numerical solutions for exponential DSE





- Recall ODE is $\gamma + \gamma\alpha\partial_\alpha\gamma = \alpha$ instead of $\gamma + \gamma(1 + w\alpha\partial_\alpha)\gamma = \alpha$.
- Resembles $w > 0$ case, but not equal to any w.
- Locus of $\partial_\alpha\gamma = 0$ no longer given by linear DSE (red), but by trivial DSE (green).

Transseries parameters for 6D model

- Linearizing in $\gamma^{\text{non-pert}}(\alpha)$ results in polynomial equation, degree equals degree of ODE.
- 4D-model: First order ODE $\Rightarrow$ unique transseries parameters (as functions of w)

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- 4D-model: First order ODE $\Rightarrow$ unique transseries parameters (as functions of w)
- 6D-model: 3rd-order ODE

$$(3+\gamma(1+w\alpha\partial_\alpha))(2+\gamma(1+w\alpha\partial_\alpha))(1+\gamma(1+w\alpha\partial_\alpha))\gamma=-\alpha.$$

- 3 distinct sets of solutions as functions of w [Balduf 2023]

$$\vec{\lambda}(w)=\left(-\frac{6}{w},-\frac{12}{w},-\frac{18}{w}\right),\quad \vec{\mu}(w)=\left(-\frac{35+29w}{6w},-\frac{5+2w}{3w},-\frac{15+13w}{2w}\right)$$
$$\vec{b}^{(1)}(w)=\left(\frac{275+267w-8w^2}{6\cdot 6^2 w},\frac{-265-624w-359w^2}{3\cdot 6^2 w},\frac{-85-241w-156w^2}{2\cdot 6^2 w}\right).$$

- Fluctuations $b^{(k)}$ around leading instanton (largest λ) are subleading corrections to perturbative asymptotics.
- Exponentials don't directly correspond to a full non-perturbative solution due to *resonance*: The λ are always integer multiples of each other. Analyzed in [Borinsky, Dunne, and Meynig 2021; Borinsky and Broadhurst 2022].

A simplified toy model that isn't simpler

- Rational Mellin transform appears to be coincidence for the 1-loop single-insertion massless multiedges.
- Consider Kreimer's toy model [Connes and Kreimer 1999; Panzer 2012]

$$\mathcal{F}\Big[B_+[t]\Big](s) := \int_0^\infty \mathrm{d}x \frac{x^{-\epsilon}}{x+s} \mathcal{F}[t](x).$$

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- Mellin transform

$$F(\rho) = \int_0^\infty \mathrm{d}x \, \frac{x^\rho}{x+1} = \frac{-\pi}{\sin(\pi\rho)} = -\frac{1}{\rho} \exp\left(\sum_{n=1}^\infty \zeta(2n) \frac{\rho^{2n}}{n} \right) = -\frac{1}{\rho} - \frac{\pi^2}{6}\rho - \frac{7\pi^4}{360}\rho^3 + \ldots .$$

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- Same DSE as always, $X = 1 + \alpha B_+[X^{1+w}]$. ODE is a pseudo differential equation

$$\frac{\sin(\pi u)}{\pi u}\Big|_{u\to\gamma(1+w\alpha\partial_\alpha)}\gamma(\alpha) = -\alpha.$$

- Empirically: $\mu(w) = -\frac{2+w}{w}$, Stokes constant $S(-2) = \frac{\pi}{2}$, some other values [Balduf 2023].

Conclusions for single-insertion DSEs

- Whenever we know the Mellin transform of the kernel graph, we can immediately write down a pseudo differential equation that determines $\gamma(\alpha)$ perturbatively (in the MOM scheme) [Balduf 2024].
- Have exact solutions for 4D model for the some choices w, among them the physically relevant $w = -2$ [Broadhurst and Kreimer 2001; Yeats 2008].
- Excellent understanding of the resurgence behavior for the 4D and 6D models [Borinsky, Dunne, and Meynig 2021; Borinsky and Dunne 2020; Borinsky and Broadhurst 2022].

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- Excellent understanding of the resurgence behavior for the 4D and 6D models [Borinsky, Dunne, and Meynig 2021; Borinsky and Dunne 2020; Borinsky and Broadhurst 2022].
- Variations:
  - Changing the parameter w changes numerical values of resurgence parameters e.g. location and nature of Borel poles, ...
  - Parameters are discontinuous when crossing $w = 0$, otherwise fairly smooth.
  - Changing the kernel Feynman rules changes the ODE, non-rational case makes systematic analysis harder, but has only little influence on factorial growth of asymptotic series (?)
  - Exponential DSE behaves qualitatively similar to positive w case.

Multiple insertion places

- Consider a single kernel, but $E \geq 1$ distinct insertion places. The DSE is then [Kreimer and Yeats 2006; Yeats 2008; Nabergall 2022; Olson-Harris 2024]

$$\gamma(\alpha) = 1 + \alpha G^{1+w_1}(\alpha, \partial_{\rho_1}) \cdots G^{1+w_E}(\alpha, \partial_{\rho_E}) \bar{F}(\rho_1, \ldots, \rho_E)\Big|_{\vec{\rho}=\vec{0}}.$$

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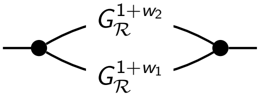
- In the combinatorial DSE, one needs to distinguish the places (i.e. use B_+ with multiple arguments) [Olson-Harris 2024].
- No longer possible to write explicit ODE for $\gamma(\alpha)$, but can use $\gamma_k = \frac{1}{k}\gamma(\alpha)\mathcal{D}\gamma_{k-1}(\alpha)$:

$$G_{\mathcal{R}}(\alpha, L) = 1 + \sum_{k=1}^{\infty} \gamma_k(\alpha) L^k = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} L^k (\gamma(\alpha)\mathcal{D})^k = e^{L\gamma(\alpha)\mathcal{D}} = 1 + \frac{e^{L\gamma\mathcal{D}} - 1}{\mathcal{D}} \gamma(\alpha).$$

(Recall $\mathcal{D} = 1 + w\alpha\partial_\alpha$ or $\mathcal{D} = \alpha\partial_\alpha$)

Example: Insertion into $E = 2$ places

$$G_{\mathcal{R}} = 1 + \alpha(1 - \mathcal{R})$$



- Series expansion of Mellin transform

$$(\rho_1 + \rho_2) \cdot F(\rho_1, \rho_2) = \bar{F}(\rho_1, \rho_2) =: \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} f_{n_1,n_2} \rho_1^{n_1} \rho_2^{n_2}.$$

- Expand all series to obtain ODE explicitly:

$$\gamma = 1 - \alpha\left(1 + \frac{e^{\partial_{\rho_1}\gamma\mathcal{D}} - 1}{\mathcal{D}}\gamma(\alpha)\right)\left(1 + \frac{e^{\partial_{\rho_2}\gamma\mathcal{D}} - 1}{\mathcal{D}}\gamma(\alpha)\right)\bar{F}(\rho)$$

$$= 1 - \alpha\left(1 + \gamma\partial_{\rho_1} + \frac{1}{2}\gamma\mathcal{D}\gamma\partial_{\rho_1}^2 + \ldots\right)\left(1 + \gamma\partial_{\rho_2} + \frac{1}{2}\gamma\mathcal{D}\gamma\partial_{\rho_2}^2 + \ldots\right)\bar{F}(\rho_1, \rho_2)$$

$$= 1 - \alpha f_{0,0} + \alpha(f_{0,1} + f_{1,0})\gamma + \alpha(f_{2,0} + f_{0,2})\mathcal{D}\gamma + \alpha f_{1,1}\gamma^2 + \alpha(f_{3,0} + f_{0,3})\gamma\mathcal{D}\gamma\mathcal{D}\gamma + \alpha(f_{2,1} + f_{1,2})\gamma^2\mathcal{D}\gamma + \mathcal{O}(\gamma^4)$$

- This gives *an* ODE, but not necessarily the best one (consider example $\bar{F}(\rho) = \frac{1}{1-\rho}$).

Double insertion asymptotics

- Growth encoded by quantity $P(\alpha) := \gamma_1 + 2\gamma_2 = \gamma - \gamma \mathcal{D} \gamma$.
  [Yeats 2008; van Baalen et al. 2009; van Baalen et al. 2010]

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- Growth encoded by quantity $P(\alpha) := \gamma_1 + 2\gamma_2 = \gamma - \gamma\mathcal{D}\gamma$.
  [Yeats 2008; van Baalen et al. 2009; van Baalen et al. 2010]
- Extensive work on the double insertion DSE, approximate differential equations, and Borel plane formulations [Bellon and Schaposnik 2008; Bellon 2010a; Bellon 2010b; Bellon and Schaposnik 2013; Bellon and Clavier 2014; Bellon and Clavier 2015; Bellon and Clavier 2017; Bellon and Russo 2021a; Bellon and Russo 2021b]
  Idea: Formally

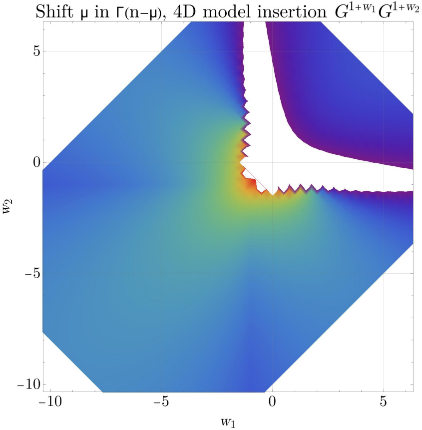
$$e^{\gamma\mathcal{D}\partial_\rho}\bar{F}(\rho)\Big|_{\rho=0} = \bar{F}(\gamma\mathcal{D}).$$

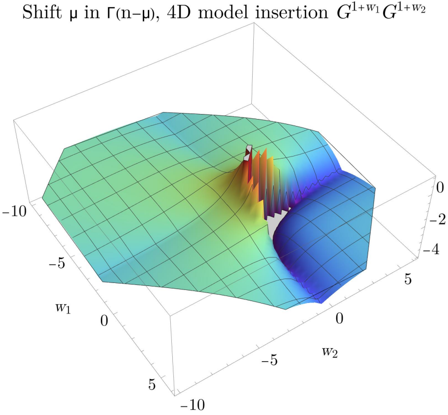
- Then do partial fraction decomposition, introduce auxiliary power series for $\frac{1}{\gamma\mathcal{D}-k} = \sum(\frac{1}{k}\gamma\mathcal{D})^n$, truncate after leading poles of $\bar{F}(\rho_1, \rho_2, \ldots)$, obtain coupled ODEs.
- For double insertion in 4D (Wess-Zumino model), $w_1 = w_2 = -2$:

$$c_n \sim S \cdot (-3)^n \Gamma\left(n + \frac{2}{3}\right).$$

(compare single insertion $w = -2$ has $(-2)^n\Gamma\left(n + \frac{1}{2}\right)$, and $w = -3$ has $(-3)^n\Gamma(n+1)$).

Asymptotic growth parameters of double insertion DSE





- Setting $w_1 = -1$ means “no insertion here”, reproduces single-insertion case.
- Function looks fairly smooth except for $w = 0$

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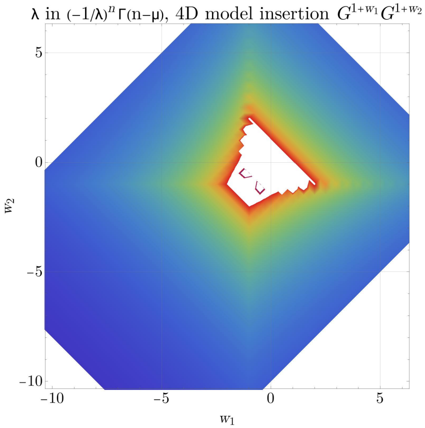

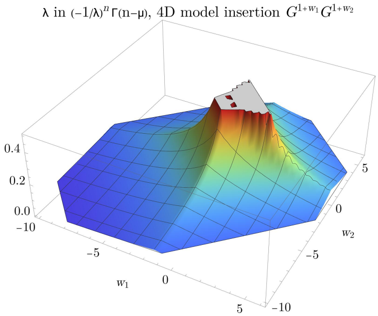

- Setting $w_1 = -1$ means “no insertion here”, reproduces single-insertion case.
- Again, case $w_j = 0$ is regular unless $w_1 = w_2 = 0$.

Conclusion for multiple insertion places

- Can still obtain ODE, but defined "implicitly".
- Systematic analysis requires study of poles of Mellin transform, partial results exist, but far less systematic than 4D and 6D model single insertion.
- Numerically, growth parameters change mildly when second insertion is included.
- Non-smooth behavior near $w_j = 0$ or $w_j = -1$.

Renormalization schemes

- Recall the counter term $S^{\mathcal{F}}_{\mathcal{R}}[X] := \mathcal{R}\mathcal{F}S[X]$.
- A *renormalization scheme* is a choice of renormalization operator $\mathcal{R}$ such that
  - The Rota-Baxter equation is fulfilled,

$$\mathcal{R}\Big(f(x)g(x)\Big) + \mathcal{R}(f(x))\mathcal{R}(g(x)) = \mathcal{R}\Big(\mathcal{R}(f(x))g(x)\Big) + \mathcal{R}\Big(f(x)\mathcal{R}(g(x))\Big),$$

  - and for every primitive graph Γ, the renormalized Feynman rules $(\mathrm{id} - \mathcal{R})\mathcal{F}[\Gamma](L)$ are finite.

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  - and for every primitive graph Γ, the renormalized Feynman rules $(\mathrm{id} - \mathcal{R})\mathcal{F}[\Gamma](L)$ are finite.
- Most notably, *Kinematic renormalization* (MOM) amounts to setting $\mathcal{R} := \mathcal{F}(X)(L=0)$ for all $X \neq \mathbb{1}$, such that $\mathcal{F}_{\mathcal{R}}(X)(L=0) = \mathbb{1}\tilde{\mathbb{1}}$.
- When working in a regularization scheme that has a regulator ϵ, such that for primitive divergent graphs $\mathcal{F}[X]$ has a pole in ϵ, then *Minimal subtraction* (MS) is defined as projection to pole parts, $\mathcal{R}' := \mathcal{F}(X)|_{\text{only pole terms in } \epsilon}$.

A more algebraic perspective

- Let $\mathcal{R}$ be kinematic renormalization, i.e. evaluation at $L=0$.
  By definition $\mathcal{F}_\mathcal{R}[X](L=0)=0$ unless $X=\mathbb{1}$.
- Let $\mathcal{R}'$ be any renormalization scheme. Define $\tau : \mathcal{H} \to \mathbb{R}$ as "extraction of the value at $L=0$", namely

$$\tau[X]=\mathcal{F}_{\mathcal{R}'}[X](L=0)=\mathcal{R}\circ\mathcal{F}_{\mathcal{R}'}[X], \qquad \tau[\mathbb{1}]=1.$$

- Non-kinematic renormalized Feynman rules are still multiplicative with respect to graph products, $\mathcal{F}_{\mathcal{R}'}[X_1\cdot X_2](L)=\mathcal{F}_{\mathcal{R}'}[X_1](L)\cdot\mathcal{F}_{\mathcal{R}'}[X_2](L)$ (follows from Rota-Baxter).
- τ is a character, because $\mathcal{F}_{\mathcal{R}'}$ is. τ and σ together determine $\mathcal{F}_{\mathcal{R}'}$:

$$\mathcal{F}_{\mathcal{R}'}[X](L)=\tau\star e^{\star L\sigma}[X].$$

- Hence, no longer $\star$-multiplicative under change of scale:

$$\mathcal{F}_{\mathcal{R}'}[X](L_1+L_2)=(\mathcal{F}_{\mathcal{R}'}(L_1)\star\mathcal{F}_{\mathcal{R}}(L_2))[X]\neq(\mathcal{F}_{\mathcal{R}'}(L_1)\star\mathcal{F}_{\mathcal{R}'}(L_2))[X].$$

Properties of MOM and MS

- Consider DimReg in both cases for easy comparison, $L = \ln \frac{p^2}{s_0}$ as always.
- In MOM with renormalization point $L = 0$
  - $G_{\mathcal{R}}(\alpha, \epsilon, L = 0) = 1$. (for all ϵ)
  - $\gamma(\alpha) = \partial_L G_{\mathcal{R}}(\alpha, \epsilon, L)\big|_{L=0}$, similar for β.
  - $\beta(\alpha, \epsilon)$ and $\gamma(\alpha, \epsilon)$ depend on ϵ
- In MS:
  - Counter terms are *only* poles in ϵ, no finite parts.
  - $\beta'(\alpha, \epsilon) = \beta(\alpha)$ and $\gamma'(\alpha, \epsilon) = \gamma(\alpha)$ (for all ϵ)
  - $G_{\mathcal{R}}(\alpha, \epsilon, L = 0) = \gamma_0(\alpha, \epsilon)$ a priori unknown
- The first coefficients of γ and γ' agree, analogous for β and β', leading counter term pole, leading log coefficient (all these quantities are determined by *period* of the kernel graph).

Choosing the right definition of the renormalization group functions

In **MOM**, the renormalization group functions $\beta(\alpha)$ and $\gamma(\alpha)$ simultaneously satisfy:

1. They are the L-derivative of $G_{\mathcal{R}}$ or $\mathcal{Q}_{\mathcal{R}}$ at $L = 0$.
2. They are the coefficients in the Callan-Symanzik equation.
3. If $Q = G_{\mathcal{R}}^{w}$ in the DSE, then $\beta = w\alpha\gamma$.
4. The beta function is the derivative of the renormalized coupling $\alpha(\alpha_0)$ with respect to the reference scale s_0 at fixed α_0.
5. The Z-factors are integrals of the renormalization group functions, or equivalently, β and γ are derivatives of the Z-factors.

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$\Rightarrow$ Define β and γ from the Z-factors to get consistent properties in all schemes. In DimReg, this produces $\gamma(\alpha, \epsilon)$, finite as $\epsilon \to 0$.

Definition

In dimensional regularization, and for all renormalization schemes, the ϵ-dependent renormalization group functions are defined as derivatives of the counterterms Z:

$$\beta'(\alpha,\epsilon) := \frac{-\epsilon}{\partial_\alpha \ln(\alpha \cdot Z_\alpha(\alpha,\epsilon))} + \alpha\epsilon$$
$$\gamma'(\alpha,\epsilon) := -(\beta'(\alpha,\epsilon) - \alpha\epsilon)\partial_\alpha \ln Z(\alpha,\epsilon).$$

- These functions satisfy, in all renormalization schemes and also for $\epsilon \neq 0$,

$$\frac{\partial}{\partial L} G_{\mathcal{R}}(\alpha,\epsilon,L) = \left(\gamma(\alpha,\epsilon) + (\beta(\alpha,\epsilon) - \alpha\epsilon)\frac{\partial}{\partial\alpha}\right) G_{\mathcal{R}}(\alpha,\epsilon,L).$$

Renormalization schemes for DSE solutions

- We have two power series:
  1. Anomalous dimension γ. In MOM, conicides with infinitesimal Feynman rule $\sigma(G_\mathcal{R}) = \partial_L G_\mathcal{R}\big|_{L=0} = \gamma_1(\alpha,\epsilon)$.
  2. Evaluation $\tau(G_\mathcal{R}) = G_\mathcal{R}\big|_{L=0} = \gamma_0(\alpha,\epsilon)$. In MOM, is constant unity.

$$\gamma_1(\alpha,\epsilon) = \gamma(\alpha,\epsilon)\mathcal{D}\gamma_0(\alpha,\epsilon).$$

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- Conceptually, the DSE is a complicated relation between these two functions. A renormalization scheme is an (arbitrary) choice of either one of these power series.
- Instead of $\gamma_0(\alpha,\epsilon)$, can equivalently use $\delta(\alpha,\epsilon)$ defined as $\tau = \exp^\star(\delta\sigma)$. Is a shift of L:

$$\mathcal{F}_{\mathcal{R}'}(L) = \tau \star e^{\star\sigma L} = e^{\star\delta\sigma} \star e^{\star L\sigma} = e^{\star(L+\delta)\sigma}.$$

- Thm: In perturbation theory, *any* renormalization scheme coincides with MOM, where the renormalization point is not $L=0$ but $L=-\delta(\alpha,\epsilon)$ [Balduf 2023]. (This is only guaranteed to work if the RGE holds.)

Shifted RGE functions

- "Hopf-algebraic" equations are perfectly concrete: Let γ be anomalous dimension in MOM, and γ' in another scheme, related by shift δ, then [Balduf 2023]

$$\gamma'(\alpha) = \frac{\gamma_1'(\alpha)}{\gamma"_0(\alpha) + w\alpha\partial_\alpha \gamma_0'(\alpha)}, \qquad \frac{\gamma(\alpha)}{\gamma'(\alpha)} = 1 + w\gamma(\alpha)\alpha\partial_\alpha\delta(\alpha).$$

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- In particular: solutions of linear DSEs
  - have the same anomalous dimension in all schemes at $\epsilon = 0$
  - are multiplied with an overall L-independent function $\gamma_0(\alpha)$ where

$$\gamma'(\alpha,\epsilon) = \gamma(\alpha,\epsilon) + \epsilon\partial_\alpha \ln \gamma_0(\alpha,\epsilon).$$

  - MS-scheme: $\gamma'(\alpha,\epsilon) = \gamma(\alpha)$ independent of $\epsilon \Rightarrow$ can infer $[\epsilon^0]\gamma'_0(\alpha,\epsilon)$ of MS from $[\epsilon^1]\gamma(\alpha,\epsilon)$ of MOM
  - Exact solution of linear DSE in MS in 4D model, where $\gamma = \frac{1}{2}(\sqrt{1+4\alpha} - 1)$ [Balduf 2024]

$$\ln \gamma'_0(\alpha) = \ln \frac{\gamma}{\alpha} - \frac{1}{4}\ln(1+4\alpha) - 2\gamma\gamma_E + \ln \frac{\Gamma(1-\gamma)}{\Gamma(1+\gamma)}, \qquad \delta(\alpha) = \frac{\ln \gamma'_0}{\gamma}.$$

Nonlinear 4D model in MS

- No exact solution known in MS for nonlinear DSE $w \neq 0$.
- Can compute shift function $\delta = \sum_{j=0}^{\infty} d_j \alpha^j$ to order ≈ 30. E.g. for 4D model, $w = -2$

$$\delta(\alpha) = -2 - \frac{3}{2}\alpha - \frac{29}{6}\alpha^2 - \left(\frac{94}{3} - \frac{1}{3}\zeta(3)\right)\alpha^3 - \ldots .$$

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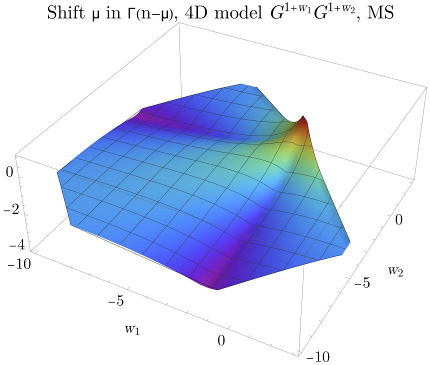
- Find empirically

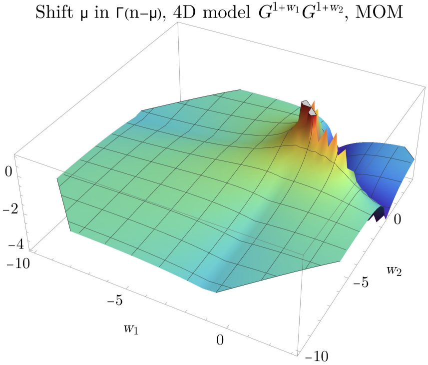
$$d_n \sim wS(w) \cdot (-w)^n \cdot \Gamma(n - \mu(w) - 1).$$

Here, $S(w), \mu(w)$ are the same coefficients as for anomalous dimension $\gamma(\alpha)$, i.e. $\frac{-d_n}{c_{n+1}} \sim 1$.

- Can determine many more parameters [Balduf 2024], upshot: $\gamma'(\alpha)$ in MS, or equivalently shift of renormalization point $\delta(\alpha)$, are asymptotic power series very similar to $\gamma(\alpha)$ in MOM.

Double insertion DSE in MS vs MOM





- Shown is the asymptotic growth of the function $\gamma_1(\alpha)$ in both schemes.
- Similar shape, but not identical numerical values.

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- If the Mellin transform is known, we can generate the ODE algorithmically. When inserting into only one place, we can write the ODE down in closed form.
- Resurgence analysis and overall good understanding of the physically sensible case $w = -2$ for the $D = 4$ and $D = 6$ models.

Conclusion 2: What have we learned from the "variations"?

- Changing the exponent w in $X = 1 + \alpha B_+ \left[X^{1+w} \right]$ changes all resurgence parameters (location and types of poles in Borel plane), but smoothly unless $w = 0$.
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- Inserting into the second edge makes ODE much more complicated, involves $\zeta(n)$, ODE is of infinite order. Still, leading resurgence parameters are mostly continuous in exponents w_1, w_2 and take similar values to single-insertion.
- Change of renormalization scheme is equivalent to shifting $L \to L - \delta(\alpha, \epsilon)$. Of the two functions $\gamma(\alpha, \epsilon)$ and $\delta(\alpha, \epsilon)$, one can be chosen freely.
- The non-linear DSEs have distinct renormalization group functions in distinct schemes. Qualitatively, MOM and MS are very similar: both divergent power series with similar factorial growth.
- In particular, the series can not be made convergent by change of scheme.

$\Rightarrow$ qualitative features of the solution (not necessarily of the methods) are relatively stable under "variations".

Thank you!

By the way, Mastodon is a social network
like Twitter, but open-source and
with elephants instead of birds!

@paulbalduf@**math**stodon.xyz

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