Amplitude Workshop, 6 December 2024
Mathematical Institute, University of Oxford

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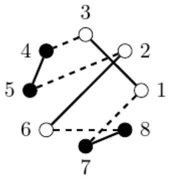
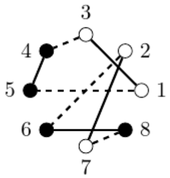
# Combinatorial proof of a non-renormalization theorem




Paul-Hermann Balduf
Postdoc, Mathematical Physics group (N1.02)

Based on arXiv 2408.03192 together with Davide Gaiotto
Slides and links are available from paulbalduf.com/research

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# How it started

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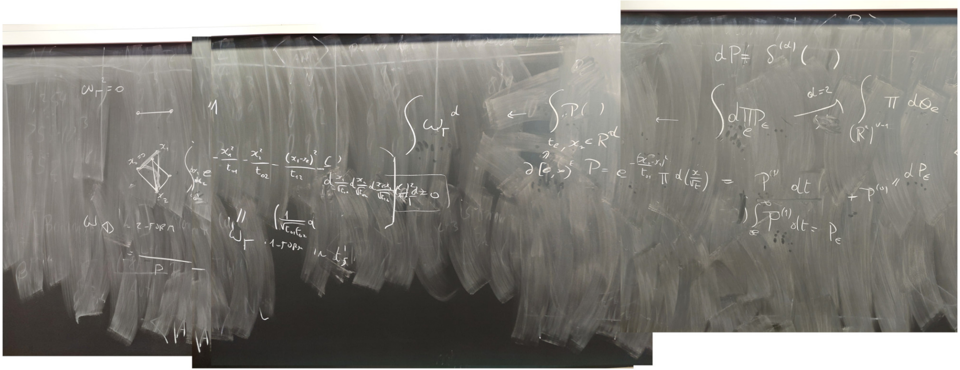
- Last year, Davide Gaiotto showed me the following combinatorial puzzle:



- There is a differential form $\alpha_\Gamma$, defined through a particular Feynman integral. Conjecture: $\alpha_\Gamma \wedge \alpha_\Gamma = 0$. Someone who likes combinatorics should prove it.

# Topological Feynman integrand

- Let $\Gamma$ be an (arbitrary) finite graph, edges $E_\Gamma$, vertices $V_\Gamma$.
- Schwinger parameter $a_e$ for each edge. Start/end coordinates $x_e^\pm \in \mathbb{R}$. Let

$$s_e := \frac{x_e^+ - x_e^-}{\sqrt{a_e}} \qquad \text{(this is a scalar, not a vector).}$$

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$$s_e := \frac{x_e^+ - x_e^-}{\sqrt{a_e}} \qquad \text{(this is a scalar, not a vector).}$$

- We are given the following parametric Feynman integral [Gaiotto, Kulp, Jingxiang Wu[1] 2024]: $\mathcal{F}(\Gamma) = \int \alpha_\Gamma$. Parametric integrand

$$\alpha_\Gamma := \int_{\{x\} \in \mathbb{R}^{|V_\Gamma|-1}} \cdots \int \bigwedge_{e \in E_\Gamma} e^{-s_e^2} \, \mathrm{d}s_e .$$

- Our job:
  1. Find a concrete, efficient formula for $\alpha_\Gamma$.
  2. Study its algebraic properties.

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# Background: Deformation quantisation

- Given is a *classical field theory*: Smooth manifold $M$. Field variable $\phi(t, \mathbf{x})$, canonical conjugate $\pi(t, \mathbf{x})$ are smooth functions on $M$. Hamilton function $H(\phi(t, \mathbf{x}), \pi(t, \mathbf{x}))$. Skew-symmetric *Poisson bracket* $\{f, g\} \in C^\infty(M)$. Gives equations of motion:

$$\partial_t \phi = \{\phi, H\}, \qquad \partial_t \pi = \{\pi, H\}, \qquad \{\phi, \pi\} = 1.$$

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- Naive quantisation: Replace $\{f,g\}$ by $\frac{i}{\hbar}\left[\hat{f},\hat{g}\right]$. Runs into inconsistencies for powers of fields. Deformation quantisation: Find a "star product" $\star$ such that

$$[f,g]_\star := f \star g - g \star f \overset{!}{=} \hbar\{f,g\} + \mathcal{O}\left(\hbar^2\right).$$

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- Power series ansatz with (to be determined) differential operators $B_j(f, g)$.

$$f \star g = B_0(f, g) + \hbar B_1(f, g) + \hbar^2 B_2(f, g) + \ldots,$$

Clearly $B_0(f, g) = f \cdot g$ and $B_1(f, g) = \frac{1}{2}\{f, g\}$. What are the higher $B_j$?

- Two conditions:
  1. Should be associative $f \star (g \star h) = (f \star g) \star h$,
  2. Should be invariant under diffeomorphisms $f \mapsto f + \hbar D_1(f) + \hbar^2 D_2(f) + \ldots$.

# Background: Deformation quantisation 2



- Solution in [Kontsevich 2003]: Consider graphs $\Gamma$ embedded in the upper half plane $\mathbb{H} = \{z \in \mathbb{C} | \Im(z) > 0\}$ with hyperbolic metric.
- In $\Gamma$, each vertex with 2 outgoing edges corresponds to a factor $\omega^{ij}\partial_i\partial_j$. (i.e. a graph $\Gamma$ encodes a nesting of Poisson brackets, a differential operator $B_\Gamma$). Graph has $n$ upper vertices and 2 vertices at bottom line $\mathbb{R}$, corresponding to arguments $f, g$ of $B_n(f, g)$.

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- Define angle $\phi(p,q)$ between geodesic $p \longrightarrow q$ and vertical line $p \longrightarrow i\infty$.
- Each graph is weighted by a weight integral $W_\Gamma = \text{const} \times \int \bigwedge_{e \in E_\Gamma} \mathrm{d}\phi_e$. Star product is (details omitted)

$$\star = \cdot \ + \sum_{n=1}^{\infty} \hbar^n \sum_{\Gamma} W_\Gamma B_\Gamma .$$

# Background: Deformation quantisation 3

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- Crucial step: Show that the so-defined $\star$ is associative.
- Associativity condition at order $\hbar^n$,

$$\sum_{k=0}^{n} B_k(B_{n-k}(f,g),h) = \sum_{k=0}^{n} B_k(f, B_{n-k}(g,h)),$$

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- Obstructions to associativity are given by certain integrals over the boundary of configuration space,

$$c_\Gamma = \int_{\partial \bar{C}_{n,m}} \bigwedge_{e \in E_\Gamma} \mathrm{d}\phi_e .$$

Since $\bigwedge_e \mathrm{d}\phi_e$ is closed, this integral vanishes and $\star$ is associative. More general, abstract statement: "Formality theorem".

# The significance of $\alpha_\Gamma \wedge \alpha_\Gamma$

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- $\alpha_\Gamma$ is a differential form in $\mathrm{d}a_e$, to be integrated over Schwinger parameters $a_e$. Itself, $\alpha_\Gamma$ is an Integral over vertex positions $x_v$ of some integrand $W_\Gamma$. Schematically:

$$\mathcal{F}(\Gamma) = \int_{\{a_e\}} \alpha_\Gamma = \int_{\{a_e\}} \int_{\{x_v\}} W_\Gamma, \qquad W_\Gamma = \bigwedge_{e \in E_\Gamma} e^{-s_e^2} \mathrm{d}s_e.$$

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$$\mathcal{F}(\Gamma) = \int_{\{a_e\}} \int_{\left\{x_v^{(1)}\right\}} \int_{\left\{x_v^{(2)}\right\}} W_\Gamma^{(1)} \wedge W_\Gamma^{(2)} = \int_{\{a_e\}} \alpha_\Gamma \wedge \alpha_\Gamma$$

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- Here, $\alpha_\Gamma \wedge \alpha_\Gamma$ is some differential form in the $\mathrm{d}a_e$'s, independent of the $x_v$. Conversely, we can exchange the order of integration and do the $\mathrm{d}a_e$ integral first. The integrand is

$$W_\Gamma^{(1)} \wedge W_\Gamma^{(2)} = \exp\left( - \sum_e \left( {s_e^{(1)}}^2 + {s_e^{(2)}}^2 \right) \right) \bigwedge_e \mathrm{d}s_e^{(1)} \wedge \mathrm{d}s_e^{(2)}.$$

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- This expression factorizes for edges. Consider an edge $e$ from point $(0,0)$ to $(x^{(1)}, x^{(2)})$:

$$e^{-{s_e^{(1)}}^2 - {s_e^{(2)}}^2} \mathrm{d}s_e^{(1)} \wedge \mathrm{d}s_e^{(2)} = e^{-\frac{\vec{x}^2}{a}} \left( -2a_e^{-2} \mathrm{d}a_e \left( x^{(2)} \mathrm{d}x^{(1)} - x^{(1)} \mathrm{d}x^{(2)} \right) + a_e^{-1} \mathrm{d}x^{(1)} \wedge \mathrm{d}x^{(2)} \right).$$

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- Only the term $\propto \mathrm{d}a_e$ contributes to integral. Polar coordinates in the plane:

$$\vec{x} = \begin{pmatrix} x^{(1)} \\ x^{(2)} \end{pmatrix} = r\begin{pmatrix} \sin\phi \\ -\cos\phi \end{pmatrix}, \qquad \frac{\mathrm{d}\vec{x}}{\mathrm{d}\phi} = r\begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} = \begin{pmatrix} -x^{(2)} \\ x^{(1)} \end{pmatrix}.$$

$\Rightarrow$ $x^{(1)}\mathrm{d}x^{(2)} - x^{(2)}\mathrm{d}x^{(1)} = \left((-x^{(2)})^2 + (x^{(1)})^2\right)\mathrm{d}\phi = |\vec{x}|^2\mathrm{d}\phi$ is the differential of the 2D angle $\phi$ of the vector $\vec{x}$.

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- Integrate the Schwinger parameter $a_e$ for a single edge:

$$\int_{a_e=0}^{\infty} e^{-{s^{(1)}}^2 - {s^{(2)}}^2} \mathrm{d}s^{(1)} \wedge \mathrm{d}s^{(2)} = \int_{a_e=0}^{\infty} e^{-\frac{|\vec{x}|^2}{a_e}} 2a_e^{-2} |\vec{x}|^2 \mathrm{d}\phi_e \wedge \mathrm{d}a_e = 2\mathrm{d}\phi_e.$$

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- We conclude that the 2-dimensional integral is (very schematically)

$$\mathcal{F}(\Gamma) = \int_{\left\{x_v^{(1)}\right\}} \int_{\left\{x_v^{(2)}\right\}} \int_{\{a_e\}} W_\Gamma^{(1)} \wedge W_\Gamma^{(2)} = \int_{\{\text{relative positions } \vec{x}_v\}} \bigwedge_e \mathrm{d}\phi_e .$$

Closer investigation of the last integral shows: These are the Kontsevich integrals $c_\Gamma$ which vanish (to make the star product associative and establish the *formality theorem*).

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  Closer investigation of the last integral shows: These are the Kontsevich integrals $c_\Gamma$ which vanish (to make the star product associative and establish the *formality theorem*).
- On the other hand:

$$c_\Gamma = \mathcal{F}(\Gamma) = \int_{\{a_e\}} \int_{\left\{x_v^{(1)}\right\}} \int_{\left\{x_v^{(2)}\right\}} W_\Gamma^{(1)} \wedge W_\Gamma^{(2)} = \int_{\{a_e\}} \alpha_\Gamma \wedge \alpha_\Gamma$$

- Hence $\int_{\{a_e\}} \alpha_\Gamma \wedge \alpha_\Gamma = 0$ implies the vanishing of Kontsevich integrals. Hence our goals:
  1. Find a concrete, efficient formula for $\alpha_\Gamma$.
  2. Show that $\alpha_\Gamma \wedge \alpha_\Gamma \equiv 0$ for all graphs except trees.

# Introducing graph matrices

- The remainder of the talk is purely combinatorial (and technical). Physical significance of $\alpha_\Gamma$ or interpretation of $\alpha_\Gamma \wedge \alpha_\Gamma$ plays no role for the construction.

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$$s_e := \frac{x_e^+ - x_e^-}{\sqrt{a_e}}, \qquad \alpha_\Gamma := \int_{\mathbb{R}^{|V_\Gamma|-1}} \cdots \int \bigwedge_{e \in E_\Gamma} e^{-s_e^2} \, \mathrm{d}s_e .$$

- Have $|E_\Gamma|$ edges, make vector $\vec{s}$. Fix one vertex $v_{|V_\Gamma|} = v_\star$ such that $x_{v_\star} = 0$. Collect remaining $(|V_\Gamma| - 1)$ vertex coordinates into $\vec{x} \in \mathbb{R}^{|V_\Gamma|-1}$.

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- Incidence matrix $\mathbb{I}$, edge matrix $\mathbb{D} = \operatorname{diag}(a_1, \ldots, a_{|E_\Gamma|})$

$$s_e = \frac{1}{\sqrt{a_e}} (\mathbb{I}\vec{x})_e, \qquad \vec{s} := \mathbb{D}^{-\frac{1}{2}} \mathbb{I}\vec{x}.$$

# Example: The dunce's cap

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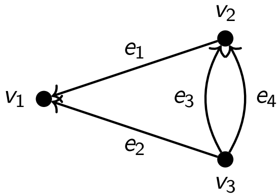
Recurring example $\Gamma$ is a graph on 3 vertices and 4 edges, with 2 loops. Labels and directions are arbitrary, we choose:

$\Gamma =$




We further choose $v_3 =: v_\star$ as the vertex to remove from $\vec{x}$.

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$\vec{x}$ has 2 entries, $\vec{s}$ has 4 entries.

$\mathbb{I}$ is $4 \times 2$ and $\mathbb{D}$ is $4 \times 4$.

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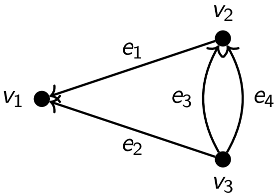


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With these choices:

$$\mathbb{I} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \mathbb{D} = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_4 \end{pmatrix}.$$

This gives for $\vec{s} = \mathbb{D}^{-\frac{1}{2}} \overline{\mathbb{I}} \vec{x}$:

$$\vec{s} = \begin{pmatrix} \frac{1}{\sqrt{a_1}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{a_2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{a_3}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{a_4}} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{x_1 - x_2}{\sqrt{a_1}} \\ \frac{x_1}{\sqrt{a_2}} \\ \frac{x_2}{\sqrt{a_3}} \\ \frac{x_2}{\sqrt{a_4}} \end{pmatrix}.$$

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# Consequences of vectorized form $\vec{s} = \mathbb{D}^{-\frac{1}{2}}\mathbb{I}\vec{x}$

- Consider the exponential in the integrand $\bigwedge_e \exp(-s_e^2)\mathrm{d}s_e = \exp\left(-\sum_e s_e^2\right)\bigwedge_e \mathrm{d}s_e$:

$$\sum_{e\in E_\Gamma} s_e^2 = \vec{s}\cdot\vec{s} = \left(\mathbb{D}^{-\frac{1}{2}}\mathbb{I}\vec{x}\right)^T\left(\mathbb{D}^{\frac{1}{2}}\mathbb{I}\vec{x}\right) = \vec{x}^T\mathbb{I}^T\mathbb{D}^{-1}\mathbb{I}\vec{x} = \vec{x}^T\mathbb{L}\vec{x}.$$

The $(|V_\Gamma|-1)\times(|V_\Gamma|-1)$ matrix $\mathbb{L} = \mathbb{I}^T\mathbb{D}^{-1}\mathbb{I}$ is the *Laplacian*.

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- For the Dunce's cap

$$\mathbb{L} = \mathbb{I}^T\mathbb{D}^{-1}\mathbb{I} = \begin{pmatrix} \frac{1}{a_1}+\frac{1}{a_2} & -\frac{1}{a_1} \\ -\frac{1}{a_1} & \frac{1}{a_1}+\frac{1}{a_3}+\frac{1}{a_4}\end{pmatrix} \qquad \vec{s}^{\,2} = \vec{x}^T\mathbb{L}\vec{x} = \frac{(x_1-x_2)^2}{a_1}+\frac{x_1^2}{a_2}+\frac{x_2^2}{a_3}+\frac{x_2^2}{a_4}.$$

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  The $(|V_\Gamma|-1)\times(|V_\Gamma|-1)$ matrix $\mathbb{L} = \mathbb{I}^T\mathbb{D}^{-1}\mathbb{I}$ is the *Laplacian*.
- For the Dunce's cap

$$\mathbb{L} = \mathbb{I}^T\mathbb{D}^{-1}\mathbb{I} = \begin{pmatrix} \frac{1}{a_1}+\frac{1}{a_2} & -\frac{1}{a_1} \\ -\frac{1}{a_1} & \frac{1}{a_1}+\frac{1}{a_3}+\frac{1}{a_4}\end{pmatrix} \qquad \vec{s}^{\,2} = \vec{x}^T\mathbb{L}\vec{x} = \frac{(x_1-x_2)^2}{a_1}+\frac{x_1^2}{a_2}+\frac{x_2^2}{a_3}+\frac{x_2^2}{a_4}.$$

- The differential $\mathrm{d}s_e$ consists of two summands:

$$\mathrm{d}s_e = \mathrm{d}\left(a_e^{-\frac{1}{2}}\left(x_{v_e^+}-x_{v_e^-}\right)\right) = -\frac{1}{2}a_e^{-\frac{3}{2}}\left(x_{v_e^+}-x_{v_e^-}\right)\mathrm{d}a_e + a_e^{-\frac{1}{2}}\left(\mathrm{d}x_{v_e^+}-\mathrm{d}x_{v_e^-}\right).$$

  Written as vectors, with $\mathrm{d}\mathbb{D} = \mathrm{diag}(\mathrm{d}a_1,\ldots,\mathrm{d}a_{|E_\Gamma|})$:

$$\mathrm{d}\vec{s} = -\frac{1}{2}\mathbb{D}^{-\frac{3}{2}}\,\mathrm{d}\mathbb{D}\,\mathbb{I}\vec{x} \;+\; \mathbb{D}^{-\frac{1}{2}}\mathbb{I}\,\mathrm{d}\vec{x}.$$

# Integrand in matrix-vector form



- Remove trivial factors: $\vec{\rho} := -2\mathbb{D}^{\frac{3}{2}} \mathrm{d}\vec{s}$,

$$\rho_e = -2a_e^{\frac{3}{2}} \mathrm{d}s_e = \left(\mathbb{I}\vec{x}\right)_e \mathrm{d}a_e - 2a_e\left(\mathbb{I}\, \mathrm{d}\vec{x}\right)_e.$$

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- Integrand involves wedge product of all $\mathrm{d}s_e$, this becomes

$$\rho_1 \wedge \rho_2 \wedge \ldots \wedge \rho_{|E_\Gamma|} = \bigwedge_{e \in E_\Gamma} \rho_e =: \bar{P}_\Gamma .$$

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$$\rho_1 \wedge \rho_2 \wedge \ldots \wedge \rho_{|E_\Gamma|} = \bigwedge_{e\in E_\Gamma} \rho_e =: \bar{P}_\Gamma.$$

- Plug in that $\prod_e \exp(-s_e^2) = \exp(-\sum_e s_e^2) = \exp(-\vec{x}^T \mathbb{L}\vec{x})$. All in all:

$$\alpha_\Gamma = \frac{1}{(-2)^{|E_\Gamma|} \prod_e a_e^{\frac{3}{2}}} \int_{\mathbb{R}^{|V_\Gamma|-1}} \cdots \int e^{-\vec{x}^T \mathbb{L}\vec{x}}\, \bar{P}_\Gamma,$$

The integration is over vertex coordinates $x_v \in \mathbb{R}$.
The resulting $\alpha_\Gamma$ is a differential form in (some of the) $\mathrm{d}a_e$.

# Counting differentials

- So far, we have

$$\alpha_\Gamma = \frac{1}{(-2)^{|E_\Gamma|} \prod_e a_e^{\frac{3}{2}}} \int_{\mathbb{R}^{|V_\Gamma|-1}} \cdots \int e^{-\vec{x}^T \mathbb{L} \vec{x}} \; \bar{P}_\Gamma,$$

$$\bar{P}_\Gamma = \big(\mathrm{d}a_1 (\mathbb{I}\vec{x})_1 - 2a_1 (\mathbb{I}\mathrm{d}\vec{x})_1\big) \; \wedge \ldots \wedge \; \big(\mathrm{d}a_{|E_\Gamma|} (\mathbb{I}\vec{x})_{|E_\Gamma|} - 2a_{|E_\Gamma|} (\mathbb{I}\mathrm{d}\vec{x})_{|E_\Gamma|}\big).$$

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- To make sense of the integral, exactly $(|V_\Gamma|-1)$ factors contribute their $\mathrm{d}x_v$ part, and the remaining $|E_\Gamma| - (|V_\Gamma|-1) = L_\Gamma$ factors contribute their $\mathrm{d}a_e$. "Loop number"
$\Rightarrow$ the form $\alpha_\Gamma$ is a differential form of degree $L_\Gamma$ in $\mathrm{d}a_e$'s.

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- To make sense of the integral, exactly $(|V_\Gamma| - 1)$ factors contribute their $\mathrm{d}x_v$ part, and the remaining $|E_\Gamma| - (|V_\Gamma| - 1) = L_\Gamma$ factors contribute their $\mathrm{d}a_e$. "Loop number" $\Rightarrow$ the form $\alpha_\Gamma$ is a differential form of degree $L_\Gamma$ in $\mathrm{d}a_e$'s.
- The relevant terms arise from selecting a set $T \subseteq E_\Gamma$ of $(|V_\Gamma| - 1)$ edges to contribute their $(-2a_e(\mathbb{I}\mathrm{d}\vec{x})_e)$, while the remaining edges contribute their $\mathrm{d}a_e(\mathbb{I}\vec{x})_e$:

$$P_\Gamma := \bar{P}_\Gamma \Big|_{\text{only terms containing } (|V_\Gamma| - 1) \text{ vertex differentials } \mathrm{d}x_v}$$

$$= \sum_{\substack{T \subseteq E_\Gamma \\ |T| = |V_\Gamma| - 1}} \operatorname{sgn}(T) \left( \prod_{e \notin T} (\mathbb{I}\vec{x})_e \right) \left( \prod_{e \in T} -2a_e \right) \left( \bigwedge_{e \notin T} \mathrm{d}a_e \right) \left( \bigwedge_{e \in T} (\mathbb{I}\mathrm{d}\vec{x})_e \right).$$

# Graphical interpretation of the terms

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$$P_\Gamma = \sum_{\substack{T\subseteq E_\Gamma \\ |T|=|V_\Gamma|-1}} \operatorname{sgn}(T) \left( \prod_{e\notin T} (\mathbb{I}\vec{x})_e \right) \left( \prod_{e\in T} -2a_e \right) \left( \bigwedge_{e\notin T} \mathrm{d}a_e \right) \left( \bigwedge_{e\in T} (\mathbb{I}\mathrm{d}\vec{x})_e \right).$$

- Each factor $\mathrm{d}x_v$ arises from some $(-2a_e(\mathbb{I}\mathrm{d}\vec{x})_e)$, corresponding to an edge $e$. Every $\mathrm{d}x_v$ (except $v_\star$) must be present once.

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University of Oxford Mathematical Institute

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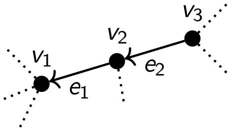
- A priori, $(\mathbb{I}\mathrm{d}\vec{x})$ contains one or two summands. A fixed $T$ could potentially give rise to multiple summands in $P_\Gamma$.
- For example, consider two edges $e_1 \in T$ and $e_2 \in T$ in some graph,




$$(\mathbb{I}\mathrm{d}\vec{x})_1 \wedge (\mathbb{I}\mathrm{d}\vec{x})_2 = (\mathrm{d}x_1 - \mathrm{d}x_2)\wedge(\mathrm{d}x_2 - \mathrm{d}x_3)$$
$$= \mathrm{d}x_1\wedge\mathrm{d}x_2 - \mathrm{d}x_1\wedge\mathrm{d}x_3 + \mathrm{d}x_2\wedge\mathrm{d}x_3.$$

(these are 3 summands, arising from one $T$)

# Graphical interpretation of the terms

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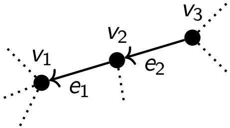
- For $e_1 \in T$ and $e_2 \in T$, there are 3 summands:

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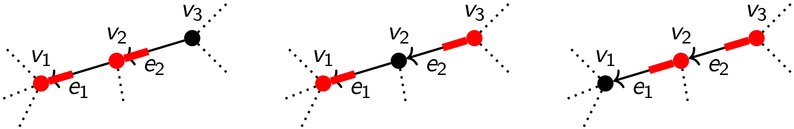
- Each summand amounts to selecting one end vertex for every edge $e \in T$:




$\mathrm{d}x_1 \wedge \mathrm{d}x_2$ $\mathrm{d}x_1 \wedge \mathrm{d}x_3$ $\mathrm{d}x_2 \wedge \mathrm{d}x_3$

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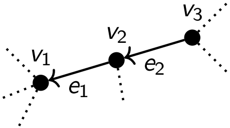


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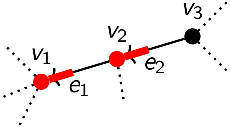
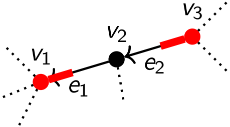
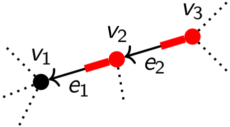



$\mathrm{d}x_1 \wedge \mathrm{d}x_2$




$\mathrm{d}x_1 \wedge \mathrm{d}x_3$




$\mathrm{d}x_2 \wedge \mathrm{d}x_3$

- There is a sign, coming from $\mathbb{I}\mathrm{d}\vec{x}$ (more later).
- What happens if the edges in $T$ contain a cycle?

# Selecting edges in a cycle

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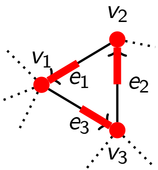
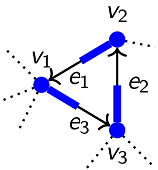
- Consider cycle of length 3 in $\{e_1, e_2, e_3\} \in T$. We have two choices of selecting which of the edges contributes which of the $\mathrm{d}x_v$:



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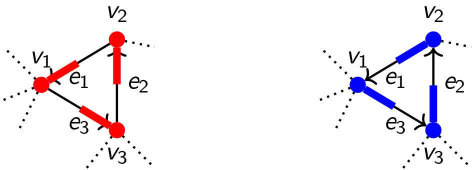


- The factors $(\mathbb{I}\mathrm{d}\vec{x})_e$ in $\alpha_\Gamma$ are ordered with respect to the edge index $e$. Here

$$(\mathbb{I}\mathrm{d}\vec{x})_1 \wedge (\mathbb{I}\mathrm{d}\vec{x})_2 \wedge (\mathbb{I}\mathrm{d}\vec{x})_3 = (\mathrm{d}x_1 - \mathrm{d}x_2) \wedge (\mathrm{d}x_2 - \mathrm{d}x_3) \wedge (\mathrm{d}x_3 - \mathrm{d}x_1).$$

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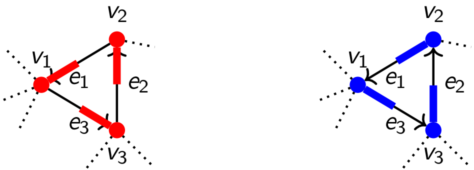


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- The two choices correspond to

$$(\mathrm{d}x_1 - \mathrm{d}x_2) \wedge (\mathrm{d}x_2 - \mathrm{d}x_3) \wedge (\mathrm{d}x_3 - \mathrm{d}x_1) = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3$$

$$(\mathrm{d}x_1 - \mathrm{d}x_2) \wedge (\mathrm{d}x_2 - \mathrm{d}x_3) \wedge (\mathrm{d}x_3 - \mathrm{d}x_1) = (-1)^3 \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_1 = -\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3$$

- Lemma: Whenever $T$ contains a cycle, its contributions cancel.
  $\Rightarrow$ sufficient to consider only such $T$ that don't contain cycles.

# Integrand as sum of trees

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- $T$ has no cycles, consists of $|V_\Gamma|-1$ edges, and adjacent to every vertex $\Rightarrow$ $T$ is connected (and has no cycles), hence $T$ is a spanning tree:

$$P_\Gamma = \sum_{\text{spanning trees } T\subseteq E_\Gamma} \operatorname{sgn}(T) \left( \prod_{e\notin T} (\mathbb{I}\vec{x})_e \right) \left( \prod_{e\in T} -2a_e \right) \left( \bigwedge_{e\notin T} \mathrm{d}a_e \right) \left( \bigwedge_{e\in T} (\mathbb{I}\mathrm{d}\vec{x})_e \right).$$

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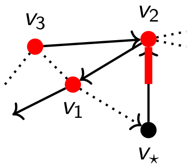
- Lemma: Each $T$ contributes exactly one term to the sum (i.e. exacly one of the two terms in $(\mathbb{I}\mathrm{d}\vec{x})_e$ for each $e \in T$). To see this, note that $v_\star$ has no $\mathrm{d}x_{v_\star}$. For every edge in $T$ that is adjacent to $v_\star$, we are bound to select the other of the two end vertices.



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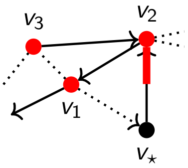


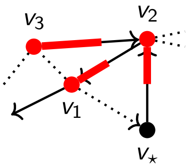


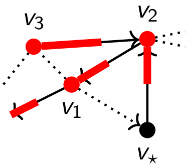

# Signs

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- What is $\operatorname{sgn}(T)$ in

$$P_\Gamma = \sum_{T \text{ spanning}} \operatorname{sgn}(T) \left( \prod_{e \notin T} (\mathbb{I}\vec{x})_e \right) \left( \prod_{e \in T} -2a_e \right) \left( \bigwedge_{e \notin T} \mathrm{d}a_e \right) \left( \bigwedge_{v \in V_\Gamma} \mathrm{d}x_v \right)?$$

- Recall that in the original definition $\bigwedge \mathrm{d}s_e$ were ordered by edge index. Each $\mathrm{d}s_e$ gives either a $\mathrm{d}a_e$ or a $\mathrm{d}x_v$.

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- There are three effects contributing to $\operatorname{sgn}(T)$:
  1. Moving all $\mathrm{d}a_e$ in front of all $\mathrm{d}x_v$.
  2. Bringing the $\mathrm{d}x_v$ into increasing order (the $\mathrm{d}a_e$ are already in increasing order).
  3. The incidence matrix in $(\mathbb{I}\mathrm{d}\vec{x})_e$ gives either $+\mathrm{d}x_v$ or $-\mathrm{d}x_v$.

# Signs

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  3. The incidence matrix in $(\mathbb{I}\mathrm{d}\vec{x})_e$ gives either $+\mathrm{d}x_v$ or $-\mathrm{d}x_v$.
- Point 1 simply gives $(-1)^{\sum_{e_j \notin T} j - \frac{L_\Gamma(L_\Gamma+1)}{2}}$.
- Points 2 and 3 are encoded by the incidence matrix. Let $\mathbb{I}[T]$ be the incidence matrix where only the rows $e \in T$ are present (this is a square matrix with entries $\pm 1$ or 0). The sign is $\det(\mathbb{I}[T])$.

# Example for signs

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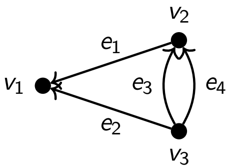


$$\mathbb{I} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$$

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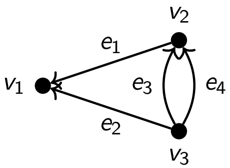


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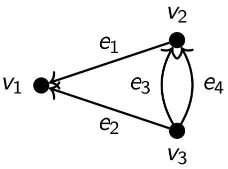
There are five spanning trees. In our particular choice of labels and direction, all of them result in $\det(\mathbb{I}[T]) = +1$, this is coincidence.

| | | | | | |
|---|---|---|---|---|---|
| $T$ : | $\{1,3\}$ | $\{1,4\}$ | $\{1,2\}$ | $\{2,3\}$ | $\{2,4\}$ |
| $\mathbb{I}[T]$ : | $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ |
| $\det\left(\mathbb{I}[T]\right)$ : | 1 | 1 | 1 | 1 | 1 |
| $(-1)^{\sum_{e_i \notin T} i - \frac{L_\Gamma}{2}}$ : | $(-1)^{2+4-1}$ | $(-1)^{2+3-1}$ | $(-1)^{3+4-1}$ | $(-1)^{1+4-1}$ | $(-1)^{1+3-1}$ |
| $\operatorname{sgn}(T)$ : | $-1$ | $+1$ | $+1$ | $+1$ | $-1$ |

# Example for signs






$$\mathbb{I} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$$

There are five spanning trees. In our particular choice of labels and direction, all of them result in $\det(\mathbb{I}[T]) = +1$, this is coincidence.

| | | | | | |
|---|---|---|---|---|---|
| $T$ : | $\{1,3\}$ | $\{1,4\}$ | $\{1,2\}$ | $\{2,3\}$ | $\{2,4\}$ |
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Plug this into the formula for $P_\Gamma$. The integrand for this graph is

$$\begin{aligned} P_\Gamma = {} & (-1) \cdot 4a_1 a_3 x_1 x_2 \cdot (\mathrm{d}a_2 \wedge \mathrm{d}a_4)(\mathrm{d}x_1 \wedge \mathrm{d}x_2) \quad + (+1) \cdot 4a_1 a_4 x_1 x_2 \cdot (\mathrm{d}a_2 \wedge \mathrm{d}a_3)(\mathrm{d}x_1 \wedge \mathrm{d}x_2) \\ & + (+1) \cdot 4a_1 a_2 x_2^2 \cdot (\mathrm{d}a_3 \wedge \mathrm{d}a_4)(\mathrm{d}x_1 \wedge \mathrm{d}x_2) \quad + (+1) \cdot 4a_2 a_3 (x_1 - x_2) x_2 \cdot (\mathrm{d}a_1 \wedge \mathrm{d}a_4)(\mathrm{d}x_1 \wedge \mathrm{d}x_2) \\ & + (-1) \cdot 4a_2 a_4 (x_1 - x_2) x_2 \cdot (\mathrm{d}a_1 \wedge \mathrm{d}a_3)(\mathrm{d}x_1 \wedge \mathrm{d}x_2). \end{aligned}$$

# $\alpha_\Gamma$ as an integral

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- We have a technical, but explicit formula:

$$P_\Gamma = \underbrace{\sum_{|T|=|V_\Gamma|-1} (-1)^{\sum_{e_i \notin T} i - \frac{L}{2}} \det\left(\mathbb{I}[T]\right) \left(\prod_{e\in T} -2a_e\right)\left(\prod_{e\notin T} (\mathbb{I}\vec{x})_e\right)\left(\bigwedge_{e\notin T} \mathrm{d}a_e\right)}_{=: W_\Gamma(\vec{x})} \bigwedge_{v\in V_\Gamma \setminus \{v_\star\}} \mathrm{d}x_v .$$

- We still need to do the actual integral over $\mathrm{d}x_v$,

$$\alpha_\Gamma := \frac{1}{(-2)^{|E_\Gamma|}\prod_e a_e^{\frac{3}{2}}} \int_{\mathbb{R}^{|V_\Gamma|-1}} \cdots \int e^{-\vec{x}^T \mathbb{L}\vec{x}}\, W_\Gamma(\vec{x}) \bigwedge_{v\in V_\Gamma\setminus\{v_\star\}} \mathrm{d}x_v .$$

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- Spanning trees have nice combinatorial properties. This implies lemmas:
  1. If $\Gamma$ is a tree ($L=0$), then $\alpha_\Gamma$ is a number, not a differential form.
  2. If the loop number $L$ is odd, then $\alpha_\Gamma = 0$.
  3. If $\Gamma$ is disconnected, then $\alpha_\Gamma = 0$.
  4. If $\Gamma = \Gamma_1 \circ \Gamma_2$ is 1-edge or 1-vertex connected, then $\alpha_\Gamma$ factorizes into $\alpha_{\Gamma_1} \wedge \alpha_{\Gamma_2}$.

  $\Rightarrow$ from now on, $\Gamma$ is 1PI (=2-vertex-connected) with even loop number.

# Standard formula for Gaussian integral

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- Standard formula for solving the vector-valued Gaussian integral:

$$\int \cdots \int e^{-\vec{x}^T \mathbb{L} \vec{x}} \, W_\Gamma(\vec{x}) \, \mathrm{d}^n \vec{x} = \frac{\pi^{\frac{n}{2}}}{\sqrt{\det \mathbb{L}}} \exp\left( \frac{1}{4} \left(\mathbb{L}^{-1}\right)_{jk} \partial_{x_j} \partial_{x_k} \right) W_\Gamma(\vec{x}) \Big|_{\vec{x}=\vec{0}}.$$

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- Denominator: $\det \mathbb{L} \prod_e a_e = \psi$ is the Symanzik polynomial. For example, the dunce's cap has

$$\psi = a_1 a_3 + a_2 a_3 + a_1 a_4 + a_2 a_4 + a_3 a_4.$$

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- Exponential function of $\partial_{x_j} \partial_{x_k}$ has only one non-vanishing term since $W_\Gamma$ is homogeneous of degree $L$ (which is even):

$$\frac{1}{\left(\frac{L}{2}\right)!} \left( \frac{1}{4} \left(\mathbb{L}^{-1}\right)_{jk} \partial_{x_j} \partial_{x_k} \right)^{\frac{L}{2}} = \frac{1}{2^L \left(\frac{L}{2}\right)!} \left( \left(\mathbb{L}^{-1}\right)_{jk} \partial_{x_j} \partial_{x_k} \right)^{\frac{L}{2}}.$$

- We need the entries $\left(\mathbb{L}^{-1}\right)_{j,k}$ of the inverse Laplacian, where $j, k$ are indices of vertices.

# Dodgson polynomials



- We need the entries $\left(\mathbb{L}^{-1}\right)_{i,j}$ of the inverse Laplacian, where $i, j$ are indices of vertices.
- Consider the *expanded Laplacian*, defined as the block matrix

$$\mathbb{M} := \begin{pmatrix} \mathbb{D} & \mathbb{I} \\ -\mathbb{I}^T & 0 \end{pmatrix}.$$

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- Let $\mathbb{M}(A, B)$ be $\mathbb{M}$ with rows $A$ and columns $B$ removed. If $|A| = |B|$, this is a square matrix, and its determinant is called *Dodgson polynomial*

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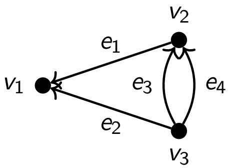
- $\mathbb{M}$ has block form, so $\mathbb{M}^{-1}$ has block form. Bottom right block is $\mathbb{L}^{-1}$. $\Rightarrow$ Lemma:

$$\left(\mathbb{L}^{-1}\right)_{i,j} = (-1)^{i+j}\frac{\psi^{i,j}}{\psi} \quad \text{(where } i, j \text{ are indices of vertices).}$$

# Example: Dogson polynomials




$$\mathbb{I} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \qquad \mathbb{M} = \begin{pmatrix} a_1 & 0 & 0 & 0 & 1 & -1 \\ 0 & a_2 & 0 & 0 & 1 & 0 \\ 0 & 0 & a_3 & 0 & 0 & 1 \\ 0 & 0 & 0 & a_4 & 0 & 1 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 & 0 \end{pmatrix}$$

In $\mathbb{M}$, the first 4 rows and columns refer to edges, the last 2 rows and columns refer to vertices $v_1, v_2$.

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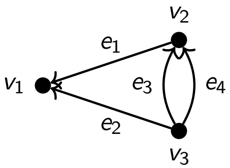


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In $\mathbb{M}$, the first 4 rows and columns refer to edges, the last 2 rows and columns refer to vertices $v_1, v_2$. Compute vertex-indexed Dodgson polynomials explicitly:

$$\psi^{v_1,v_1} = \det \begin{pmatrix} a_1 & 0 & 0 & 0 & -1 \\ 0 & a_2 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 1 \\ 0 & 0 & 0 & a_4 & 1 \\ 1 & 0 & -1 & -1 & 0 \end{pmatrix} = a_2(a_1 a_3 + a_1 a_4 + a_3 a_4)$$

$$\psi^{v_1,v_2} = -a_2 a_3 a_4 = \psi^{v_2,v_1}, \qquad \psi^{v_2,v_2} = (a_1 + a_2) a_3 a_4.$$

Indeed,

$$\mathbb{L}^{-1} = \frac{1}{a_3 a_4 + a_1 a_3 + a_1 a_4 + a_2 a_3 + a_2 a_4} \begin{pmatrix} a_2(a_3 a_4 + a_1(a_3 + a_4)) & a_2 a_3 a_4 \\ a_2 a_3 a_4 & (a_1 + a_2) a_3 a_4 \end{pmatrix}.$$

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# Wick's theorem

- Using $\psi^{i,j}$ for the entries of $\mathbb{L}^{-1}$, we now have

$$\alpha_\Gamma = \frac{\pi^{\frac{|V_\Gamma|-1}{2}}}{2^{L+|E_\Gamma|}\left(\frac{L}{2}\right)! \cdot \prod_e a_e \cdot \psi^{\frac{L+1}{2}}} \left( \sum_{i,j \in V_\Gamma} (-1)^{i+j} \psi^{j,k} \partial_{x_j} \partial_{x_k} \right)^{\frac{L}{2}} W_\Gamma(\vec{x}).$$

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- $W_\Gamma$ is a sum over spanning trees. Extract the contribution of a single $T$ by

$$[\mathrm{d}\bar{T}]\alpha_\Gamma := \left[\bigwedge_{e \notin T} \mathrm{d}a_e\right] \alpha_\Gamma = \alpha_\Gamma \Big|_{\text{Terms proportional to } \bigwedge_{e\notin T} \mathrm{d}a_e}.$$

The corresponding polynomial in the integrand $W_\Gamma$ is

$$X_T := \prod_{e \notin T} (\mathbb{I}\vec{x})_e = \sum_{s=1}^{\leq 2^L} X_{T,s}.$$

Each $X_{T,s}$ is a monomial of degree $L$ in the $x_v$. Sign $\operatorname{sgn}(X_{T,s})$ from signs in $\mathbb{I}$.

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Each $X_{T,s}$ is a monomial of degree $L$ in the $x_v$. Sign $\operatorname{sgn}(X_{T,s})$ from signs in $\mathbb{I}$.
- Wick's theorem: Sum over all permutations $\sigma \in S_L$ of the indices,

$$\left( (-1)^{j+k} \psi^{j,k} \partial_{x_j} \partial_{x_k} \right)^{\frac{L}{2}} X_{T,s} = \operatorname{sgn}(X_{T,s}) (-1)^{\sum_{x_k\in X_{T,s}} k} \sum_{\sigma\in S_L} \psi^{\sigma(x_{i_1}),\sigma(x_{i_2})} \cdots \psi^{\sigma(x_{i_{L-1}}),\sigma(x_{i_L})}.$$

# Dodgson identities

- We now have a fully explicit formula for the contribution of spanning tree $T$,

$$\left[\mathrm{d}\,\bar{T}\right]\alpha_\Gamma = \frac{\pi^{\frac{|V_\Gamma|-1}{2}} \operatorname{sgn}(T)(-1)^{|V_\Gamma|-1}}{4^L\left(\frac{L}{2}\right)! \cdot \prod_{e\notin T} a_e \cdot \psi^{\frac{L+1}{2}}}$$

$$\cdot \sum_s \operatorname{sgn}(X_{T,s})(-1)^{\sum_{x_k\in X_{T,s}} k} \sum_{\sigma\in S_L} \psi^{\sigma(x_{i_1}),\sigma(x_{i_2})} \cdots \psi^{\sigma(x_{i_{L-1}}),\sigma(x_{i_L})}.$$

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- There is still unused information: The monomials $X_{T,s}$ arise from one and the same $\prod_{e \notin T} (\mathbb{I}\vec{x})_e$, that is, they involve the same $x_v$. Phrased differently: The sum $\sum_s X_{t,s} = (x_+ - x_-)_{e_i}(x^+ - x^-)_{e_j} \cdots$ factorizes.

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- The vertex Dodgson polynomials factorize in the same way! Lemma: Let $e_1 = v_1 \to v_2$ and $e_2 = v_3 \to v_4$ be distinct edges, then

$$(-1)^{v_2+v_4}\psi^{v_2,v_4} - (-1)^{v_1+v_4}\psi^{v_1,v_4} + (-1)^{v_1+v_3}\psi^{v_1,v_3} - (-1)^{v_2+v_3}\psi^{v_2,v_3}$$
$$= (-1)^{e_1+e_2+1} a_{e_1} a_{e_2} \cdot \psi^{e_1,e_2}.$$

- $\Rightarrow$ The sum over all monomials $X_{T,s}$ of a fixed $T$ collapses to a single product of *edge-indexed* Dodgson polynomials

# The end result



**Theorem** [PB, Gaiotto 2024]: $\alpha_\Gamma = \sum_{\text{spanning tree } T} \left[\mathrm{d}\bar{T}\right] \alpha_\Gamma \cdot \bigwedge_{e \notin T} \mathrm{d}a_e$, where

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Here, $\psi^{i,j}$ are edge-indexed Dodgson polynomials and the sum goes over all permutations $\sigma$ of the $L$ edges in $\bar{T}$, and these edges are assumed to be in increasing order as always.

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Comments:

1. We can access coefficients of individual $\bigwedge_{e\notin T}\mathrm{d}a_e$ without computing everything else.
2. The remaining sum depends only on the edges $\bar{T}$ (i.e. it can be implemented easily, without first enumerating auxiliary terms). It has only $(L-1)!!$ distinct summands, i.e. 3 summands for $L=4$, 15 for $L=6$ etc.
3. $\psi$ and $\psi^{i,j}$ can be computed very quickly with standard linear algebra programs.

# Proving that it squares to zero

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- We will show that $\alpha_\Gamma \wedge \alpha_\Gamma = 0$ for all $\Gamma$ except trees.
- Notice that it is not trivially zero: $\alpha_\Gamma$ is a form of even degree $L \in 2\mathbb{N}$.

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- Notice that it is not trivially zero: $\alpha_\Gamma$ is a form of even degree $L \in 2\mathbb{N}$.
- Each $\alpha_\Gamma$ is a sum over spanning trees, $[\mathrm{d}\,\bar{T}]\alpha_\Gamma$. No $\mathrm{d}a_e$ must appear twice $\Rightarrow$ select a set $E$ of $2L$ distinct edges

$$E_\Gamma \supseteq E = \bar{T}_1 \cup \bar{T}_2.$$

- Every potentially nonvanishing term in $\alpha_\Gamma \wedge \alpha_\Gamma$ contains $2L$ differentails $\mathrm{d}a_e$ of this form,

$$\alpha_\Gamma \wedge \alpha_\Gamma = \sum_{\substack{E \subseteq E_\Gamma \\ |E|=2L}} \bar{Q}_E \, \mathrm{d}E.$$

We prove that $\bar{Q}_E = 0$ for all $E$.

# Yet another sign

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- We defined $\bar{Q}_E := [\mathrm{d}E](\alpha_\Gamma \wedge \alpha_\Gamma)$.
- In general, there are many possible ways to make the same set $E$ from a disjoint union of two spanning tree complements $E = \bar{T}_1 \cup \bar{T}_2$. In $\mathrm{d}E$, the $\mathrm{d}a_e$ are sorted. Hence, there is an additional sign depending on which edges were in $\bar{T}_1$ and which in $\bar{T}_2$.
- If we consider $E_1$ and $E_2$ as *words* (i.e. sets with a fixed order of elements), then the extra sign is

$$\operatorname{sgn}(E_1 \cup E_2) = \operatorname{sgn}_{\text{perm}}(E_1 E_2) \cdot \operatorname{sgn}_{\text{perm}}(E_1) \cdot \operatorname{sgn}_{\text{perm}}(E_2)$$

  Here $E_1 E_2$ is concatenation of the words, and the permutation sign is the usual one, e.g. $\operatorname{sgn}_{\text{perm}}(1,2,3,4) = 1, \operatorname{sgn}_{\text{perm}}(1,2,4,3) = -1$. With this

$$\bar{Q}_E = \sum_{E = E_1 \cup E_2} \operatorname{sgn}(E_1 \oplus E_2) \cdot [\mathrm{d}E_1]\alpha_\Gamma \cdot [\mathrm{d}E_2]\alpha_\Gamma .$$

- Plug in our formula for $[\mathrm{d}E_j]\alpha_\Gamma = [\mathrm{d}\bar{T}_j]\alpha_\Gamma$.

## Structure of $Q_E$

- We leave out uninteresting scalar factors. Then

$$\tilde{Q}_E := \sum_{E=E_1\oplus E_2} \operatorname{sgn}(E_1 \oplus E_2) \det\left(\mathbb{I}(E_1,\emptyset)\right) \det\left(\mathbb{I}(E_2,\emptyset)\right) \cdot \sum_{\sigma\in S_L(E_1)} \psi^{\sigma(e_1),\sigma(e_2)} \ldots \psi^{\sigma(e_{L-1}),\sigma(e_L)} \cdot \sum_{\sigma\in S_L(E_2)} \psi^{\sigma(e_1),\sigma(e_2)} \ldots \psi^{\sigma(e_{L-1}),\sigma(e_L)}.$$

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- Spend a few weeks with Dodgson polynomials to realize that

$$\{-1,0,+1\} \ni \det\left(\mathbb{I}(E_1,\emptyset)\right) \det\left(\mathbb{I}(E_2,\emptyset)\right) = \psi^{E_1,E_2}.$$

This is a Dodgson polynomial, where very many edges (not just one) have been deleted from $\mathbb{M}$. This determinant $\det(M(E_1,E_2))$ can be expanded in terms of cofactors (Jacobi's determinant formula from 1841)

$$\psi^{E_1,E_2} = \frac{1}{(\psi)^{L-1}} \sum_{m\in \text{matchings}(E_1,E_2)} \operatorname{sgn}(m) \underbrace{\prod_{(p_1,p_2)\in m} \psi^{p_1,p_2}}_{L \text{ factors}}.$$

# Auxiliary graphs

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- With this, $Q_E$ is a product of sums over edge-indexed Dodgson polynomials,

$$Q_E = \sum_{S \in S_{2L}(E)} \operatorname{sgn}_{\text{perm}}(S) \cdot \psi^{s_1, s_{L+1}} \psi^{s_2, s_{L+2}} \dots \psi^{s_L, s_{2L}} \cdot \sum_{\sigma \in S_L(E_1)} \psi^{\sigma(s_1), \sigma(s_2)} \dots \psi^{\sigma(s_{L-1}), \sigma(s_L)} \cdot \sum_{\sigma \in S_L(E_2)} \psi^{\sigma(s_{L+1}), \sigma(s_{L+2})} \dots \psi^{\sigma(s_{2L-1}), \sigma(s_{2L})}.$$

# Auxiliary graphs

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$$Q_E = \sum_{S \in S_{2L}(E)} \operatorname{sgn}_{\text{perm}}(S) \cdot \overbrace{\psi^{s_1, s_{L+1}} \psi^{s_2, s_{L+2}} \cdots \psi^{s_L, s_{2L}}}^{L \text{ dashed lines between } L \text{ black and } L \text{ white}}$$

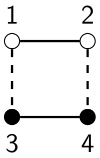
$$\cdot \sum_{\sigma \in S_L(E_1)} \underbrace{\psi^{\sigma(s_1), \sigma(s_2)} \cdots \cdot \psi^{\sigma(s_{L-1}), \sigma(s_L)}}_{\frac{L}{2} \text{ solid lines between the } L \text{ white vertices}} \cdot \sum_{\sigma \in S_L(E_2)} \underbrace{\psi^{\sigma(s_{L+1}), \sigma(s_{L+2})} \cdots \psi^{\sigma(s_{2L-1}), \sigma(s_{2L})}}_{\frac{L}{2} \text{ solid lines between the } L \text{ black vertices}}.$$

- This is confusing. Use a graphical notation. Draw an edge $e \in E_1$ as a white vertex, and $e \in E_2$ as a black vertex. The $\psi^{i,j}$ are edges between these vertices.
- In particular, the first sum consists of $L$ edges, each of which joins one black vertex to one white vertex. Draw them dashed. The other two sums are between vertices of the same color. Draw them thick.
- The first sum contains every index exactly once. Hence there is exactly one dashed line incident to every vertex. Similarly, there is exactly one solid line from the second two sums. $\Rightarrow$ our auxiliary graph is a collection of cycles with alternating edges.

# Cancelling of auxiliary graphs

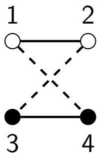
- For example, consider $E=\{1,2,3,4\}$, partitioned into $E_1=\{1,2\}$ and $E_2=\{3,4\}$. The first sum has two terms: $\psi^{1,3}\psi^{2,4}-\psi^{1,4}\psi^{2,3}$. These terms can be identified with the two permutations $(3,4)$ and $(4,3)$ of $E_2$. The second sum is just $\psi^{1,2}$, the third sum is $\psi^{3,4}$. These two terms correspond to the auxiliary graphs:

$$+(1,2)\oplus(3,4)$$




$$+\psi^{1,3}\psi^{2,4}\psi^{1,2}\psi^{3,4}$$

$$-(1,2)\oplus(4,3)$$

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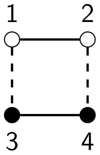


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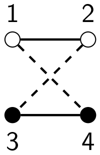
$-(1,2) \oplus (4,3)$

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- Each type of edge gives the same factor $\psi^{i,j}$. Idea: Exchange dashed and solid edges. This gives the same product of $\psi^{i,j}$s, but with opposite sign.

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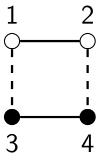


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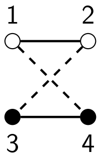


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- Each type of edge gives the same factor $\psi^{i,j}$. Idea: Exchange dashed and solid edges. This gives the same product of $\psi^{i,j}$s, but with opposite sign.
- More precisely: Pick one cycle. Fix two vertices at opposite ends, exchange all other vertices pairwise. The number of edges in a cycle is $\in 4\mathbb{N}$. Hence, this operation is an odd number of pairwise exchanges, hence an odd permutation, hence it flips the sign.

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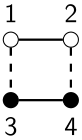
# Vanishing for 2-loop graphs

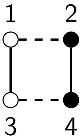
$+(1,2) \oplus (3,4)$




$+\psi^{1,3}\psi^{2,4}\psi^{1,2}\psi^{3,4}$

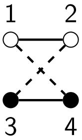
fix $\{1,4\}$
exchange $2 \leftrightarrow 3$

$-(1,3) \oplus (2,4)$




$-\psi^{1,2}\psi^{3,4}\psi^{1,3}\psi^{2,4}$

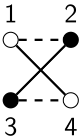
$-(1,2) \oplus (4,3)$




$-\psi^{1,4}\psi^{2,3}\psi^{1,2}\psi^{3,4}$

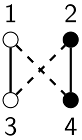
fix $\{1,3\}$
exchange $2 \leftrightarrow 4$

$+(1,4) \oplus (2,3)$




$+\psi^{1,2}\psi^{3,4}\psi^{1,4}\psi^{2,3}$

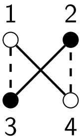
$+(1,3) \oplus (4,2)$




$+\psi^{1,4}\psi^{2,3}\psi^{1,3}\psi^{2,4}$

fix $\{1,2\}$
exchange $3 \leftrightarrow 4$

$-(1,4) \oplus (3,2)$




$-\psi^{1,3}\psi^{2,4}\psi^{1,4}\psi^{2,3}$

# Vanishing of $\alpha_\Gamma \wedge \alpha_\Gamma$

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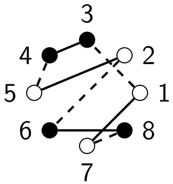
- If the auxiliary graph has multiple disjoint cycles, pick an arbitrary one and do the flip.
- For every term in the sums in $Q_E$, we can identify a term which cancels it. Often there is more than one, but they still cancel pairwise.
- **Theorem** [PHB, Gaiotto 2024]: If $\Gamma$ is not a tree, then, for any choice $E \subseteq E_\Gamma$ of $2L$ distinct edges, $Q_E = 0$. Therefore, $\alpha_\Gamma \wedge \alpha_\Gamma = 0$.

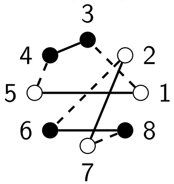
# Example for a non-connected auxiliary graph (4 loops)

$+(1,5,2,7) \oplus (3,4,6,8)$




$+\psi^{1,3}\psi^{4,5}\psi^{2,6}\psi^{7,8}\psi^{1,7}\psi^{2,5}\psi^{3,4}\psi^{6,8}$

$+(1,5,2,7) \oplus (3,4,6,8)$

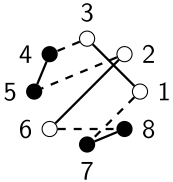



$+\psi^{1,3}\psi^{4,5}\psi^{2,6}\psi^{7,8}\psi^{1,5}\psi^{2,7}\psi^{3,4}\psi^{6,8}$

fix $\{1,2\}$
exchange $3 \leftrightarrow 7 \quad 4 \leftrightarrow 8, \quad 5 \leftrightarrow 6$

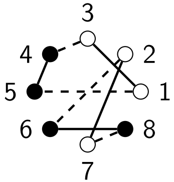
fix $\{1,4\}$
exchange $3 \leftrightarrow 5$

$-(1,6,2,3) \oplus (7,8,5,4)$




$-\psi^{1,7}\psi^{6,8}\psi^{2,5}\psi^{3,4}\psi^{1,3}\psi^{2,6}\psi^{7,8}\psi^{4,5}$

$-(1,3,2,7) \oplus (5,4,6,8)$




$-\psi^{1,5}\psi^{3,4}\psi^{2,6}\psi^{7,8}\psi^{1,3}\psi^{2,7}\psi^{4,5}\psi^{6,8}$

# Summary

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- We started from

$$s_e := \frac{x_e^+ - x_e^-}{\sqrt{a_e}}, \qquad \alpha_\Gamma := \int \cdots \int_{\mathbb{R}^{|V_\Gamma|-1}} \bigwedge_{e \in E_\Gamma} e^{-s_e^2} \, ds_e .$$

- We have found an explicit formula for $\alpha_\Gamma$, without integrals, in terms of graph polynomials that can be generated with standard linear algebra methods. Mathematica notebook is available.
- We have proved that $\alpha_\Gamma \wedge \alpha_\Gamma = 0$, which amounts to vanishing of Kontsevich integrals.
- We have also proved several new technical lemmas.

# What's that good for?

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- Proving $\alpha_\Gamma \wedge \alpha_\Gamma = 0$ was an important consistency check.

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- The topological form $\alpha_\Gamma$ has a particularly simple structure. However, Dodgson polynomials also appear e.g. in QED [Golz 2017]. $\Rightarrow$ moving towards a better graph-theoretic control of parametric Feynman integrals with numerators.

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- Explicit formula has also revealed interesting connections to the theory of canonical differential forms on graph complexes [ongoing with Simone Hu].

Thank you!

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# Scalar parametric Feynman rules

- Scalar massless Feynman integral in momentum space. Dimension $D$, loop momentum vectors $\underline{k}_l$. (Including masses and arbitrary propagator powers is no problem)

$$\mathcal{F}(\Gamma) = \left( \prod_{\ell \in L_\Gamma} \int \frac{\mathrm{d}^D \, \underline{k}_\ell}{(2\pi)^D} \right) \left( \prod_{e \in E_\Gamma} \frac{1}{\underline{k}_e^2} \right).$$

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- External momentum $\underline{\lambda}_v$ at vertex $v$. Integrate over edge momenta $\underline{k}_e$ for all edges, not just loops. Collect all momenta into vectors $\vec{k}, \vec{\lambda}$. One equation per vertex:

$$0 = \sum_{e \sim v} \underline{k}_e + \underline{\lambda}_v = \sum_{e \in E_\Gamma} \mathbb{I}_{e,v} \underline{k}_e + \underline{\lambda}_v \quad \Rightarrow \quad \mathbb{I}^T \vec{k} + \vec{\lambda} = \vec{0}.$$

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- Introduce variable $x_v$ for each vertex, write $\delta(\mathbb{I}^T \vec{k} + \vec{\lambda})$ as integral of $e^{i \vec{x}^T (\mathbb{I}\vec{k} + \vec{\lambda})}$.

$$\mathcal{F}(\Gamma) = \left( \prod_{e \in E_\Gamma} \int \frac{\mathrm{d}^D \underline{k}_e}{(2\pi)^D} \right) \left( \prod_{v \in V_\Gamma} \int_{-\infty}^{\infty} \mathrm{d}^D \underline{x}_v \right) \left( \prod_{e \in E_\Gamma} \frac{1}{\underline{k}_e^2} \right) e^{i \vec{k}^T \tilde{\mathbb{I}} \vec{x} + i \vec{\lambda}^T \vec{x}}.$$

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# Scalar parametric Feynman rules 2

- Schwinger trick (= Feynman trick = heat kernel representation)

$$\frac{1}{\underline{k}_e^2} = \int_0^\infty \mathrm{d}a_e \; e^{-a_e \underline{k}_e^2}.$$

- Feynman integral now is over vertex positions $x_v$ *and* edge momenta $k_e$ *and* Schwinger parameters $a_e$. But: Simple integrand, using $\sum_e a_e k_e^2 = \vec{k}^T \mathbb{D} \vec{k}$,

$$\mathcal{F}(\Gamma) = \left(\prod_{e\in E_\Gamma} \int_0^\infty \mathrm{d}a_e\right)\left(\prod_{e\in E_\Gamma} \int \frac{\mathrm{d}^D \underline{k}_e}{(2\pi)^D}\right)\left(\prod_{v\in (V_\Gamma \setminus v_0)} \int \mathrm{d}^D \underline{x}_v\right) \exp\Big(-\vec{k}^T \mathbb{D}\vec{k} + i\vec{k}^T \mathbb{I}\vec{x} + i\vec{x}^T \vec{\lambda}.\Big).$$

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- Solve Gaussian integral over $\vec{k}_e$:

$$\mathcal{F}(\Gamma) = \left( \prod_{e \in E_\Gamma} \int_0^\infty \mathrm{d}a_e \right) \left( \prod_{v \in (V_\Gamma \setminus v_0)} \int \mathrm{d}^D \underline{x}_v \right) \frac{1}{(\det \mathbb{D})^{\frac{D}{2}}} \exp\left( -\frac{1}{4} \vec{x}^T \mathbb{I}^T \mathbb{D}^{-1} \mathbb{I} \vec{x} + i \vec{x}^T \vec{\lambda} \right).$$

- The matrix $\mathbb{I}^T \mathbb{D}^{-1} \mathbb{I} := \mathbb{L}$ is the Laplacian.

# Scalar parametric Feynman rules 3

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- Solve Gaussian integral over $x_v$:

$$\mathcal{F}(\Gamma) = \left( \prod_{e \in E_\Gamma} \int_0^\infty \mathrm{d}a_e \right) \frac{1}{(\det \mathbb{D})^{\frac{D}{2}} (\det \mathbb{L})^{\frac{D}{2}}} e^{-\vec{\lambda}^T \mathbb{L}^{-1} \vec{\lambda}}.$$

# Scalar parametric Feynman rules 3

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- Solve Gaussian integral over $x_v$:

$$\mathcal{F}(\Gamma) = \left( \prod_{e \in E_\Gamma} \int_0^\infty \mathrm{d}a_e \right) \frac{1}{(\det \mathbb{D})^{\frac{D}{2}} (\det \mathbb{L})^{\frac{D}{2}}} e^{-\vec{\lambda}^T \mathbb{L}^{-1} \vec{\lambda}}.$$

- Introduce notation: First Symanzik polynomial $\psi = \det \mathbb{D} \det \mathbb{L}$, Second Symanzik polynomial $\phi := \psi \ \vec{\lambda}^T \mathbb{L}^{-1} \vec{\lambda}$. Parametric Feynman integral is

$$\mathcal{F}(\Gamma) = \int_0^\infty \cdots \int I_\Gamma, \qquad I_\Gamma = \frac{e^{-\frac{\phi}{\psi}}}{\psi^{\frac{D}{2}}} \prod_{e \in E_\Gamma} \mathrm{d}a_e.$$

- The differential form $\alpha_\Gamma$ is our analogue of $I_\Gamma$.

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