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Perimeter Institute

Topological Feynman integrals and the odd graph complex

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Based on arXiv 2503.09558 (Documenta Mathematica …) together with Simone Hu,
and arXiv 2408.03192 (JHEP …) together with Davide Gaiotto.
Slides and links are available from paulbalduf.com/research.

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Introduction

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$$\alpha_G = \phi_G \qquad \text{(up to constants).}$$

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1. What is the topological form α_G? What does it compute in topological QFT?

$\sim$ interlude about graph matrices $\sim$

2. What is the Pfaffian form ϕ_G? How is it used in the odd graph complex?
3. What can we learn from them being equal?

(I will not present details of the proof for $\alpha_G = \phi_G$)

TQFT Propagator $P_n(\vec{x})$

- Consider n-dimensional topological QFT, position space $\vec{x} = (x^{(1)}, \ldots, x^{(n)})^\intercal$ with field differential operator = de Rham operator

$$\mathrm{d} = \mathrm{d}x^{(1)}\partial_{x^{(1)}} + \mathrm{d}x^{(2)}\partial_{x^{(2)}} + \ldots + \mathrm{d}x^{(n)}\partial_{x^{(n)}}.$$

- Propagator is Green function of d, defined by $\mathrm{d}P_n(\vec{x}) = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\ \delta^n(\vec{x})\ \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n$.

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$$P_n(\vec{x}) = \frac{\Omega_n}{|\vec{x}|^n} = \frac{\sum_{j=1}^n (-1)^j x^{(j)}\,\mathrm{d}x^{(1)} \wedge \widehat{\mathrm{d}x^{(j)}} \wedge \mathrm{d}x^{(n)}}{\sqrt{\vec{x}\cdot\vec{x}}^{\;n}}.$$

- Ω_n is the projective n-dimensional volume form (= infinitesimal surface element in spherical coordinates). In particular

$$P_1 = \frac{x}{|x|} = \mathrm{sgn}(x), \qquad P_2 = \frac{x^{(2)}\,\mathrm{d}x^{(1)} - x^{(1)}\,\mathrm{d}x^{(2)}}{{x^{(1)}}^2 + {x^{(2)}}^2} = \frac{r^2\sin^2\phi\,\mathrm{d}\phi + r^2\cos^2\phi\,\mathrm{d}\phi}{r^2} = \mathrm{d}\phi.$$

Parametric representation of the TQFT propagator $P_n(\vec{x})$

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- Recall integral representation of Euler gamma function,

$$\frac{1}{|\vec{x}|^n} = \frac{1}{\Gamma\left(\frac{n}{2}\right)} \int_0^\infty \frac{1}{a^{\frac{n}{2}+1}} e^{-\frac{\vec{x}^2}{a}} \, \mathrm{d}a.$$

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- For each component $x^{(j)}$ introduce $s^{(j)} := \frac{x^{(j)}}{\sqrt{a}}$. Then, $\mathrm{d}s^{(j)} = \frac{\mathrm{d}x^{(j)}}{a^{\frac{1}{2}}} - \frac{x^{(j)}}{2a^{\frac{3}{2}}} \mathrm{d}a$ [Gaiotto, Kulp, and Wu 2024; Budzik et al. 2023]. Wedge product:

$$\mathrm{d}s^{(1)} \wedge \ldots \wedge \mathrm{d}s^{(n)} = \frac{\mathrm{d}x^{(1)} \wedge \ldots \wedge \mathrm{d}x^{(n)}}{a^{\frac{n}{2}}} + \frac{\mathrm{d}a \wedge \Omega_n}{2a^{\frac{n}{2}+1}}.$$

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- If one integrates a, first term vanishes, and

$$\int_0^\infty e^{-\vec{s}^2} \mathrm{d}s^{(1)} \wedge \ldots \wedge \mathrm{d}s^{(n)} = \frac{\Gamma\left(\frac{n}{2}\right)}{2} \frac{\Omega_n}{(\vec{x}^2)^{\frac{n}{2}}} = \frac{\Gamma\left(\frac{n}{2}\right)}{2} P_n(\vec{x}).$$

- Notice that the integrand factorizes: $e^{s^{(1)2}} \mathrm{d}s^{(1)} \wedge e^{s^{(2)2}} \mathrm{d}s^{(2)} \wedge e^{s^{(3)2}} \mathrm{d}s^{(3)} \wedge \ldots$.

Brackets

- Use BRST formalism: BRST differential Q such that gauge-invariant "physical" observables A are 0^{th} cohomology group. That is,

$$QA = 0 \qquad \text{and} \quad \nexists B : A = QB.$$

- A classically gauge invariant observable might violate gauge invariance at quantum level ("anomaly"). Work in perturbation theory, let $\mathcal{O}_j$ be local operators. Define *bracket* [Gaiotto, Kulp, and Wu 2024]

$$\{\mathcal{O}_1, \ldots, \mathcal{O}_k\} \coloneqq Q\left(\int_{\mathbb{R}^{n(k-1)}} \mathcal{O}_1 \cdots \mathcal{O}_k\right).$$

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$$\{\mathcal{O}_1, \ldots, \mathcal{O}_k\} := Q \left(\int_{\mathbb{R}^{n(k-1)}} \mathcal{O}_1 \cdots \mathcal{O}_k \right).$$

- The integral is a sum over Feynman integrals with k vertices in the n-dimensional TQFT,

$$\{\mathcal{O}_1, \mathcal{O}_2, \ldots\} = \sum_{\text{Graphs } G} \frac{1}{|\text{Aut}(G)|} I_G \prod_{v \in V_G} \prod_i \varphi_{i,v}.$$

symmetry factor

Feynman integral

External leg structure

The topological form

- Recall that parametric integrand factorizes along dimension ⇒ consider 1-dimensional integrand α_G. Schwinger parameter a_e for each edge. Start/end coordinates $x_e^\pm \in \mathbb{R}$. Then $I_G = \int \alpha_G \wedge \alpha_G \wedge \dots$ with the *topological form* (differential form of degree ℓ)

$$\alpha_G := \frac{1}{\pi^{\frac{|E_G|}{2}}} \int_{\mathbb{R}^{|V_G|-1}} \cdots \int \bigwedge_{e \in E_G} e^{-s_e^2} \, \mathrm{d}s_e, \qquad \text{where} \quad s_e := \frac{x_e^+ - x_e^-}{\sqrt{a_e}}.$$

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- Key results of [Balduf and Gaiotto 2024]:

$$\alpha_G = \frac{1}{\pi^{\frac{\ell}{2}} 4^\ell \left(\frac{\ell}{2}\right)! \cdot \psi_G^{\frac{\ell+1}{2}}} \sum_{\substack{T \text{ spanning} \\ \text{tree}}} \det\left(\mathbb{I}[T]\right) \left(\sum_{\sigma\in\mathfrak{S}_T} \psi_G^{\sigma(f_1),\sigma(f_2)} \cdots \psi_G^{\sigma(f_{\ell-1}),\sigma(f_\ell)}\right) \bigwedge_{f\notin T} \mathrm{d}a_f,$$

$$\alpha_G \wedge \alpha_G = 0 \quad \text{for all graph (Kontsevich Formality theorem).}$$

Here $\mathbb{I}$ is the edge-vertex incidence matrix, ψ_G is the Symanzik polynomial, ψ^{e_1,e_2} are edge-induced Dodgson polynomials (all of these can be produced easily with a computer).

Graph matrices 1: Incidence matrix and Laplacian

- Always assume that the graph G is connected. Edge set E, vertex set V.
- $|E| \times (|V|-1)$ *incidence matrix* $\mathbb{I}$ has entry $\mathbb{I}_{e,v} = +1$ if edge e ends at vertex v, and -1 if e starts at v, and 0 else. Column of one vertex $v_\star$ left out.

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- $|E| \times |E|$ *edge variable matrix* $\mathbb{D} = \operatorname{diag}\left(a_1, \ldots, a_{|E|}\right)$ contains Schwinger parameters.
- $(|V|-1) \times (|V|-1)$ *vertex Laplacian*

$$\mathbb{L} := \mathbb{I}^\intercal \mathbb{D}^{-1} \mathbb{I}.$$

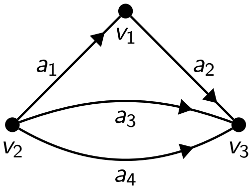
- *First Symanzik polynomial*

$$\psi_G := \det \mathbb{L} \cdot \det \mathbb{D} = \det \mathbb{L} \cdot \prod_{e \in E} a_e = \sum_{T \text{ spanning}} \prod_{e \notin T} a_e$$

is homogeneous of degree ℓ in the variables a_e.

Example: The dunce's cap

"Dunce's cap" G is a graph on 3 vertices and 4 edges, with $\ell = 2$ loops. Labels and directions are chosen as:



We further choose $v_3 =: v_\star$ as the vertex to remove from $\vec{x}$.
Remaining: $|V| = 2, |E| = 4$.
$\mathbb{I}$ is 4×2 and $\mathbb{D}$ is 4×4.

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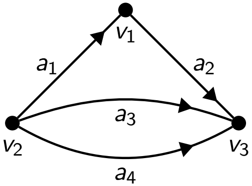

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With these choices:

$$\mathbb{I} = \begin{pmatrix} 1 & -1 \\ -1 & 0 \\ 0 & -1 \\ 0 & -1 \end{pmatrix}, \qquad \mathbb{D} = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_4 \end{pmatrix}.$$

This gives the Laplacian $\mathbb{L} = \mathbb{I}^\intercal \mathbb{D} \mathbb{I}$:

$$\mathbb{L} = \begin{pmatrix} \frac{1}{a_1} + \frac{1}{a_2} & -\frac{1}{a_1} \\ -\frac{1}{a_1} & \frac{1}{a_1} + \frac{1}{a_3} + \frac{1}{a_4} \end{pmatrix}.$$

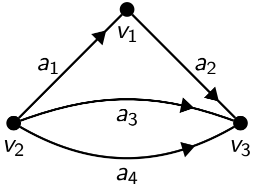
Symanzik polynomial:

$$\psi_G = \det \mathbb{L} \cdot \prod_{e \in E} a_e = a_3 a_4 + a_1(a_3 + a_4) + a_2(a_3 + a_4).$$

(Notice *matrix tree theorem*: The terms of ψ are the complements of spanning trees, $\psi = \sum_T \prod_{e \notin T} a_e$).

Topological differential form for the dunce's cap

$$\alpha_G = \frac{1}{\pi^{\frac{\ell}{2}} 4^{\ell} \left(\frac{\ell}{2}\right)! \cdot \psi_G^{\frac{\ell+1}{2}}} \sum_{\substack{T \text{ spanning} \\ \text{tree}}} \det\left(\mathbb{I}[T]\right) \left(\sum_{\sigma \in \mathfrak{S}_{\overline{T}}} \psi_G^{\sigma(f_1),\sigma(f_2)} \cdots \psi_G^{\sigma(f_{\ell-1}),\sigma(f_\ell)} \right) \bigwedge_{f \notin T} \mathrm{d}a_f \,.$$



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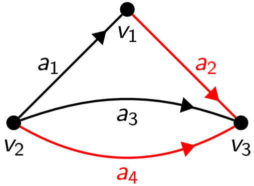

G has five spanning trees T. For example, consider $T = \{2,4\}$. Then $E \setminus T = \{f_1, f_2\} = \{1,3\}$ and $\mathbb{I}[T] = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\psi^{1,3} = -a_4$ (I didn't introduce how to compute this).
One obtains the contribution

$$\frac{(+1)}{16\pi(a_1a_3 + a_2a_3 + a_1a_4 + a_2a_4 + a_3a_4)^{3/2}} \cdot (-2a_4)\,\mathrm{d}a_1 \wedge \mathrm{d}a_3 .$$

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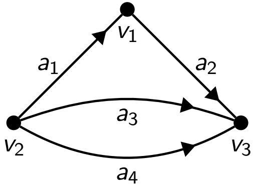

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End result:

$$\alpha_G = \frac{-a_4(\mathrm{d}a_1 \wedge \mathrm{d}a_3 + \mathrm{d}a_2 \wedge \mathrm{d}a_3) + a_3(\mathrm{d}a_1 \wedge \mathrm{d}a_4 + \mathrm{d}a_2 \wedge \mathrm{d}a_4) - (a_1 + a_2)\,\mathrm{d}a_3 \wedge \mathrm{d}a_4}{8\pi(a_1 a_3 + a_2 a_3 + a_1 a_4 + a_2 a_4 + a_3 a_4)^{3/2}} .$$

Graph matrices 2: Cycle incidence matrix

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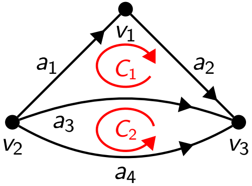
- A *circuit* is a closed path of edges (regardless of edge directions). May visit vertex, but not edge, multiple times.
- Circuits can be added and subtracted, form a vector space over $\mathbb{Z}$ (mod ± 2). *Cycle space*, dimension: $|E| - |V| + 1 = \ell$ is *loop number*.
- A choice of basis for cycle space determines a *cycle incidence matrix* $\mathcal{C}$: Entry $\mathcal{C}_{e,c} = +1$ if edge e is in cycle c in positive direction, -1 if in negative direction.
- Analogously, vertex incidence matrix $\mathbb{I}$ represents a choice of basis in *cut space*.
- The spaces, and hence the matrices $\mathcal{C}$ and $\mathbb{I}$ are orthogonal, $\mathbb{I}^\intercal \mathcal{C} = \mathbb{0}_{(|V|-1)\times \ell}$, $\mathcal{C}^\intercal \mathbb{I} = \mathbb{0}_{\ell \times (|V|-1)}$.

Example: Cycles in the dunce's cap



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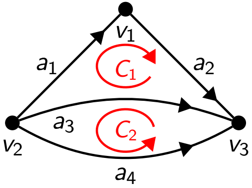

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With C_1 and C_2 as drawn,
$C_1 = \{+a_1, +a_2, -a_3\}$ and $C_2 = \{-a_3, +a_4\}$.

$$\mathcal{C} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ -1 & -1 \\ 0 & 1 \end{pmatrix}, \qquad \text{recall} \quad \mathbb{I} = \begin{pmatrix} 1 & -1 \\ -1 & 0 \\ 0 & -1 \\ 0 & -1 \end{pmatrix}.$$

Columns of $\mathcal{C}$ are basis vectors in cycle space, columns of $\mathbb{I}$ are basis vectors in cut space.
Cut space and cycle space are orthogonal, i.e.

$$\mathcal{C}^\intercal \mathbb{I} = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 0 \\ 0 & -1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Graph matrices 3: Cycle Laplacian

- Recall the vertex Laplacian $\mathbb{L} := \mathbb{I}^\intercal \mathbb{D}^{-1} \mathbb{I}$, is a $(|V|-1)\times(|V|-1)$ sym. matrix.
- Analogously *cycle Laplacian* is the $\ell \times \ell$ symmetric matrix $\mathbb{\Lambda} := \mathcal{C}^\intercal \mathbb{D} \mathcal{C}$.
- Determinant is $\det \mathbb{\Lambda} = \psi_G$ (regardless of the choice of $\mathcal{C}$). Hence, $\mathbb{\Lambda}$ is invertible.

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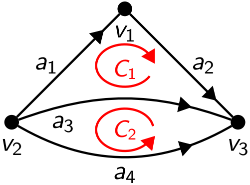

$C_1 = \{+a_1, +a_2, -a_3\}$ and $C_2 = \{-a_3, +a_4\}$.

$$\mathcal{C} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ -1 & -1 \\ 0 & 1 \end{pmatrix}, \qquad \mathbb{\Lambda} = \begin{pmatrix} a_1+a_2+a_3 & a_3 \\ a_3 & a_3+a_4 \end{pmatrix}.$$

Inverse matrix denominator is Symanzik polynomial $\det \mathbb{\Lambda} = \psi_G$,

$$\mathbb{\Lambda}^{-1} = \frac{1}{\psi_G} \begin{pmatrix} a_3+a_4 & -a_3 \\ -a_3 & a_1+a_2+a_3 \end{pmatrix}.$$

Graph matrices 4: Path matrices

- A *path matrix* $\mathcal{P}$ is a $|E| \times (|V| - 1)$–matrix where column j is a directed path of edges from $v_\star$ to v_j.
- $\mathcal{P}$ has the same shape as $\mathbb{I}$, but they are distinct. In fact, $\mathcal{P}^\intercal \mathbb{I} = \mathbb{1}_{(|V|-1)\times(|V|-1)}$.

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- One can show that $\det(\mathcal{C} \,|\, \mathcal{P}) \in \{+1, -1\}$. This determinant encodes a (relative) sign ambiguity that arises from the choice of cycle basis in $\mathcal{C}$ [Conant and Vogtmann 2003].

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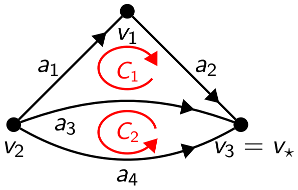

Let $v_\star = v_3$ and paths $P_1 = \{a_1, -a_3\}$ and $P_2 = \{-a_4\}$.

$$\mathcal{C} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ -1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{P} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbb{I} = \begin{pmatrix} 1 & -1 \\ -1 & 0 \\ 0 & -1 \\ 0 & -1 \end{pmatrix}.$$

The concatenation $(\mathcal{C} \,|\, \mathcal{P})$ has full rank and $\det(\mathcal{C} \,|\, \mathcal{P}) = +1$.
One also checks that $\mathcal{P}^\intercal \mathbb{I} = \mathbb{1}_{2\times 2}$.
It is coincidence that all matrices have the same shape.

Pfaffians

- Let M be a $2n \times 2n$ skew-symmetric matrix. The *Pfaffian* is

$$\operatorname{Pf}(M) = \frac{1}{2^n n!} \sum_{\sigma \in \mathfrak{S}_{2n}} \operatorname{sgn} \sigma \cdot M_{\sigma(1),\sigma(2)} \cdots M_{\sigma(2n-1),\sigma(2n)}.$$

- If a skew-symmetric M has odd dimensions, set $\operatorname{Pf}(M) = 0$.
  Then $\operatorname{Pf}(M)^2 = \det(M)$ for all skew-symmetric matrices.
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- This (like the determinant) assumes that the entries of M commute.
- Examples:

$$\operatorname{Pf}\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} = b, \qquad \operatorname{Pf}\begin{pmatrix} 0 & b & c & d \\ -b & 0 & g & h \\ -c & -g & 0 & l \\ -d & -h & -l & 0 \end{pmatrix} = bl - ch + dg.$$

The Pfaffian form

- Consider a graph with even loop number ℓ, and differential wrt Schwinger parameters

$$\mathrm{d}\mathbb{\Lambda} = \mathrm{d}\left(\mathcal{C}^{\mathsf{T}} \mathbb{D} \mathcal{C}\right) = \mathcal{C}^{\mathsf{T}}\, \mathrm{d}\mathbb{D}\mathcal{C}.$$

  Then the matrix $\mathrm{d}\mathbb{\Lambda} \cdot \mathbb{\Lambda}^{-1} \cdot \mathrm{d}\mathbb{\Lambda}$ is a $\ell \times \ell$ (=even), skew-symmetric matrix whose entries are 2-forms (hence they commute).

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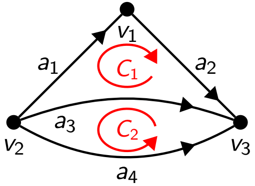
- Change of cycle basis $\mathcal{C}' = A^\intercal \mathcal{C} A$ with constant matrix A leads to

$$\mathrm{d}\mathbb{\Lambda}' \mathbb{\Lambda}'^\intercal \,\mathrm{d}\mathbb{\Lambda} = A^\intercal \,\mathrm{d}\mathbb{\Lambda} A \;\left(A^\intercal \mathbb{\Lambda} A\right)^{-1} A^\intercal \,\mathrm{d}\mathbb{\Lambda} A = A^\intercal \;\mathrm{d}\mathbb{\Lambda}\mathbb{\Lambda}^{-1}\,\mathrm{d}\mathbb{\Lambda}\; A$$

  known: $\mathrm{Pf}(A^\intercal BA) = \det(A)\,\mathrm{Pf}(B)$.

  $\Rightarrow \phi_G$ changes sign by $\det(A)$ under change of basis.

Example: Pfaffian form of the dunce's cap



$$\mathcal{C} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ -1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \Lambda^{-1} = \frac{1}{\psi_G} \begin{pmatrix} a_3 + a_4 & -a_3 \\ -a_3 & a_1 + a_2 + a_3 \end{pmatrix}$$

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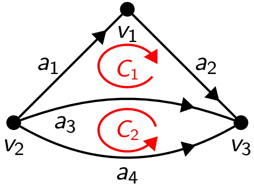

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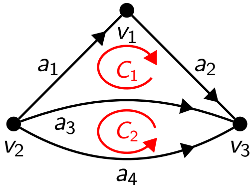

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We only need the top right entry of

$$\mathrm{d}\mathbb{\Lambda}\mathbb{\Lambda}^{-1}\,\mathrm{d}\mathbb{\Lambda} = \frac{1}{\psi_G} \begin{pmatrix} \mathrm{d}a_1 + \mathrm{d}a_2 + \mathrm{d}a_3 & \mathrm{d}a_3 \\ \mathrm{d}a_3 & \mathrm{d}a_3 + \mathrm{d}a_4 \end{pmatrix} \begin{pmatrix} (a_3 + a_4)(\mathrm{d}a_1 + \mathrm{d}a_2) + a_4\,\mathrm{d}a_3 & a_4\,\mathrm{d}a_3 - a_3\,\mathrm{d}a_4 \\ -a_3(\mathrm{d}a_1 + \mathrm{d}a_2) + (a_1 + a_2)\,\mathrm{d}a_3 & (a_1 + a_2)(\mathrm{d}a_3 + \mathrm{d}a_4) + a_3\,\mathrm{d}a_4 \end{pmatrix}$$

This yields

$$\phi_G = \frac{a_4\,\mathrm{d}a_1\,\mathrm{d}a_3 + a_4\,\mathrm{d}a_2\,\mathrm{d}a_3 - a_3\,\mathrm{d}a_1\,\mathrm{d}a_4 - a_3\,\mathrm{d}a_2\,\mathrm{d}a_4 + (a_1 + a_2)\,\mathrm{d}a_3\,\mathrm{d}a_4}{-2\pi\psi_G^{\frac{3}{2}}}.$$

The main result

Compare the two example calculations for the dunce's cap:

$$\phi_G = \frac{a_4\,\mathrm{d}a_1\,\mathrm{d}a_3 + a_4\,\mathrm{d}a_2\,\mathrm{d}a_3 - a_3\,\mathrm{d}a_1\,\mathrm{d}a_4 - a_3\,\mathrm{d}a_2\,\mathrm{d}a_4 + (a_1+a_2)\,\mathrm{d}a_3\,\mathrm{d}a_4}{-2\pi\psi_G^{\frac{3}{2}}}$$

$$\alpha_G = \frac{-a_4(\,\mathrm{d}a_1\,\mathrm{d}a_3 + \,\mathrm{d}a_2\,\mathrm{d}a_3) + a_3(\,\mathrm{d}a_1\,\mathrm{d}a_4 + \,\mathrm{d}a_2\,\mathrm{d}a_4) - (a_1+a_2)\,\mathrm{d}a_3\,\mathrm{d}a_4}{8\pi(a_1a_3 + a_2a_3 + a_1a_4 + a_2a_4 + a_3a_4)^{3/2}} = \frac{1}{4}\phi_G.$$

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Theorem (PHB and Hu 2025). Let $\mathcal{C}$ be any choice of cycle incidence matrix and $\mathcal{P}$ any choice of path matrix, then $\det(\mathcal{C}\,|\,\mathcal{P}) \in \{+1, -1\}$ and for all graphs

$$\alpha_G = \frac{\det(\mathcal{C}\,|\,\mathcal{P})}{2^\ell} \cdot \phi_G.$$

Proof: Linear algebra, expansion formulas for Pfaffians, match the Dodgson polynomial formula for the topological form α_G.

What is the Pfaffian form good for?

It acts on the odd graph complex...

The even graph complex

- Graph complexes are important combinatorial objects in math and physics.
- The *even* commutative graph complex GC_N is vector space over $\mathbb{Q}$, freely generated by (G, η) where G: connected graph without 1- or 2-valent vertices, η: orientation (=permutation sign of ordering of edges) [Kontsevich 1993].
- Grading $\deg(G) = |E| - N \cdot \ell$ (we choose $N = 2$ since then grading = sdd). All even N give isomorphic complexes.

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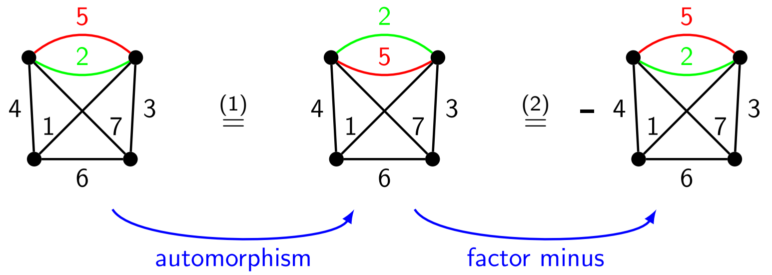
- Modulo isomorphism $f : G \mapsto \tilde{G}$ by $(G, \eta) \overset{(1)}{=} (\tilde{G}, f(\eta))$ and $-(G, \eta) \overset{(2)}{=} (G, -\eta)$. This implies that all graphs with double edges (or other odd automorphisms) vanish.

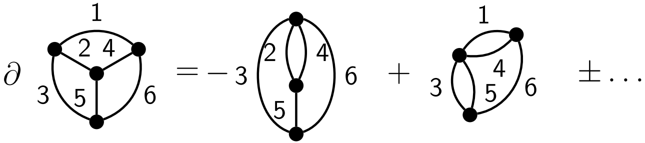


Boundary map of graph complexes

- Let G/γ denote shrinking of subgraph $\gamma \subset G$ to a vertex. Define the boundary operator

$$\partial(G,\eta) = \sum_{j=1}^{n} (-1)^j \left(G/e_j, \eta/e_j\right).$$

Example:



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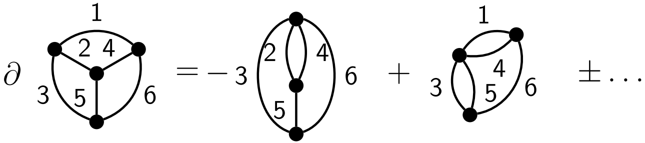

- *Graph homology* is $H_\bullet(\mathrm{GC}_N) = \ker \partial / \operatorname{im} \partial$. It is graded by (homological) degree, H_n where $n = \deg(G) = |E| - N\ell$, and by loop number .

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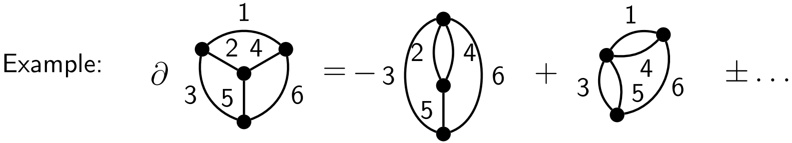

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- Example: The above graph W_3 (=wheel on 3 spokes) has $\partial W_3 = 0$ since all resulting graphs contain double edges. $\deg(W_3) = 6 - 2 \times 3 = 0$. Turns out it is not exact, $\nexists F : \partial F = W_3$. Hence $W_3 \in H_0(\mathrm{GC}_2)$.

Homology of the even graph complex

- Example: Wheel with n spokes W_n is closed $\partial W_n = 0$ since contracting any edge yields a double edge (which vanishes).
- However, $W_{2n} = 0$ due to odd automorphism. Can show that $[W_{2n+1}] \in H_0(\mathrm{GC}_2)\ \forall n \geq 1$. They all have degree 0 in GC_2 since $4n + 2 - 2(2n+1) = 0$.

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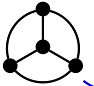
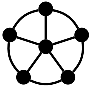
- Homologies are known up to $\ell \approx 10$ [Brun and Willwacher 2024]. One finds only few classes, but for $\ell \to \infty$, their dimension grows super-exponentially [Borinsky and Zagier 2024].

Homologies of GC_2:

H_6	vanishes due to						0	0	
H_4	2-valent vertex					0	0	0	
H_3					0	1	0	1	
H_2				0	0	0	0	0	
H_1			0	0	0	0	0	0	
H_0		0	1	0	1	0	1	1	
ℓ	1	2	3	4	5	6	7	8	...





Brown's canonical differential forms

- Let G be a connected graph with cycle Laplacian $\mathbb{\Lambda} = \mathcal{C}^\intercal \mathbb{D} \mathcal{C}$. Define *canonical form* [Brown 2021]

$$\beta_G^n := \operatorname{tr}\left(\left(\mathbb{\Lambda}^{-1} \, \mathrm{d}\mathbb{\Lambda} \right)^n \right).$$

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- If $k > 0$, the form is projectively invariant; $\mathrm{d}\beta^{4k+1} = 0$.
- These are the *primitive* canonical forms (i.e. define a coproduct Δ such that $\Delta \beta^{4k+1} = \mathbb{1} \otimes \beta^{4k+1} + \beta^{4k+1} \otimes \mathbb{1}$). They generate an algebra of canonical forms, where products might have different degree. E.g.

$$\beta^5 \wedge \beta^9 \quad \text{has degree } 14 \neq 4k+1.$$

- If ω_G is a canonical form of degree n and $|E| = n+1$, then ω is proportional to the projective volume form $\Omega_{|E|}$,

$$\omega_G = \frac{\text{some polynomial}}{\psi^j} \Omega_{|E|}.$$

Canonical integrals

- Canonical forms can be used to find cohomology classes in the graph complex.
  Let G be some (linear combination of) graphs such that $\partial G = 0$ (this can be checked by explicit computation). Hard part: How to establish whether $\exists F$ such that $\partial F = G$?

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- As $\mathrm{d}\beta = 0$, it is $\int_F \mathrm{d}\beta = 0$ for every graph F, where $\int_F = \int_{\sigma_F}$ with $\sigma_F = [a_1 : \ldots : a_{|E|}] \in \mathbb{P}(\mathbb{R}^{|E|})_+$ ("graph simplex").

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  (There are more terms for a non-primitive $\omega = \beta \wedge \beta \wedge \ldots$, but it still vanishes).
- Conversely, if one finds *any* β such that $\int_G \beta \neq 0$, one knows that $G \neq \partial F$. That is, G is not exact, and since $\partial G = 0$, this G defines a cohomology class in the even graph complex.
- Equivalently, one can view the integrals as elements of the *dual* of the complex ($I_G(\omega) := \int_G \omega$ is a linear map from GC_2 to $\mathbb{R}$).

The odd graph complex

- The *odd graph complex* consists of oriented graphs (G, η), where the orientation η is a labeling of vertices + a choice of edge directions (= the information which is contained in the incidence matrix $\mathbb{I}$, mod 2).
  η equivalent to (cycle basis + edge order) [Conant and Vogtmann 2003].
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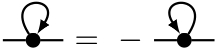
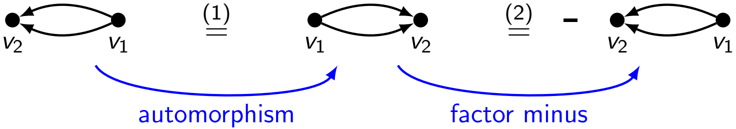
- Tadpoles vanish: 
- Multi edges no longer vanish automatically, but graphs which are *only* multi edges with even number of edges (=odd number of loops) vanish.



Homology of the odd graph complex

- Same boundary operator as for GC_2:

$$\partial(G,\eta) = \sum_{j=1}^{n} (-1)^j \left(G/e_j, \eta/e_j\right).$$

- Example: All even-loop multi edges are closed.
- H_{-3} is "algebra of 3-graphs" [Duzhin, Kaishev, and Chmutov 1998; Vogel 2011].

Homologies of GC_3:

H_{-8}		known to						0	
H_{-7}		vanish					0	0	
H_{-6}						1	1	2	
H_{-5}					0	0	0	0	
H_{-4}				0	0	0	0	0	
H_{-3}		1	1	1	2	2	3	4	
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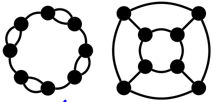

The role of the Pfaffian form

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- Recall that canonical forms β_G^{4k+1} operate on the *even* graph complex.
- The odd graph complex requires a form that flips sign in the same way as the graphs do.

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- The Pfaffian form ϕ_G has this property [Brown, Hu, and Panzer 2024], it is an "orientation form". Concretely, for a cycle Laplacian $\mathbb{\Lambda} \mapsto A^\intercal \mathbb{\Lambda} A$ we have

$$\beta^{4k+1} \mapsto \beta^{4k+1}, \qquad \text{but} \qquad \phi \mapsto \det(A)\phi.$$

$\Rightarrow \int_G \phi_G \wedge \omega$ is well-defined on the odd graph complex, where ω is any product of β forms.

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- Can use $\int_G \phi \wedge \omega$ to compute homology. But note $\Delta\phi = \phi \otimes \phi \Rightarrow$ more terms in Stokes relation.
- Example from [Brown, Hu, and Panzer 2024]: For $\ell = 6$, the form $\beta^5 \wedge \phi$ is of degree 11. There is a linear combination of graphs with $\ell = 6$ and $|E| = 12$ where the integral is non-vanishing, it spans the homology H_{-6} at $\ell = 6$.

Consequences

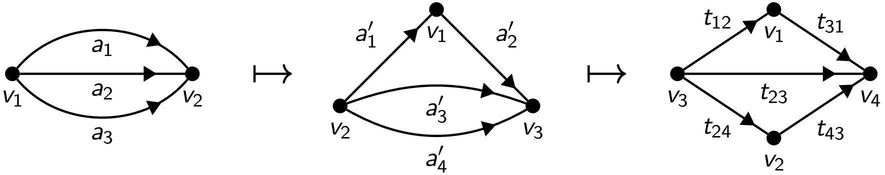
Now we know what α is and what ϕ is, and that they are the same.
What can we learn from this?

Consequences

- Obtained a new representation for α_G. Since ϕ_G is (directly) given by matrices, many of its properties follow easily from linear algebra.
- $\mathrm{d}\alpha_G = 0$, and $\int \alpha_G$ is finite, projective, well-defined under change of labelings, etc.

Consequences

- Obtained a new representation for α_G. Since ϕ_G is (directly) given by matrices, many of its properties follow easily from linear algebra.
- $\mathrm{d}\alpha_G = 0$, and $\int \alpha_G$ is finite, projective, well-defined under change of labelings, etc.
- For example: Contracting (non-tadpole) edge, or inserting 2-valent vertex into edge, is canonical isomorphism s of cycle space. Then $\alpha_{G'} = \pm s^*(\alpha_G)$.



$$\frac{a_3\,\mathrm{d}a_1\wedge\mathrm{d}a_2}{(a_1a_2+a_2a_3+a_1a_3)^{3/2}} \mapsto \frac{a'_4(\mathrm{d}a'_1+\mathrm{d}a'_2)\wedge\mathrm{d}a'_3}{\big((a'_1+a'_2)a'_3+(a'_1+a'_2)a'_4+a'_3a'_4\big)^{3/2}}$$

$$\mapsto \frac{(t_{24}+t_{43})(\mathrm{d}t_{12}+\mathrm{d}t_{31})\wedge\mathrm{d}t_{23}}{\big((t_{12}+t_{31})t_{23}+(t_{12}+t_{31})(t_{24}+t_{43})+t_{23}(t_{24}+t_{43})\big)^{3/2}}.$$

Formality theorem

- Kontsevich formality theorem [Kontsevich 2003] $\alpha_G \wedge \alpha_G = 0$ (there are no anomalies in TQFTs with $D \geq 2$) proved with some effort in [Balduf and Gaiotto 2024; Wang and Williams 2024].

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- Now use that $\mathrm{Pf}(A)^2 = \det(A)$:

$$\phi_G \wedge \phi_G \propto \frac{1}{\det \mathbb{\Lambda}} \left(\mathrm{Pf}\left(\mathrm{d}\mathbb{\Lambda}\mathbb{\Lambda}^{-1}\,\mathrm{d}\mathbb{\Lambda}\right)\right)^2 = \det\left(\mathbb{\Lambda}^{-1}\right)\det\left(\mathrm{d}\mathbb{\Lambda}\;\mathbb{\Lambda}^{-1}\,\mathrm{d}\mathbb{\Lambda}\right) = \det\left(\mathbb{\Lambda}^{-1}\,\mathrm{d}\mathbb{\Lambda}\;\mathbb{\Lambda}^{-1}\,\mathrm{d}\mathbb{\Lambda}\right)$$

$$= \det\left(\left(\mathbb{\Lambda}^{-1}\,\mathrm{d}\mathbb{\Lambda}\right)^2\right) =: \det\left(M\right) = \frac{1}{(\ell/2)!} B_n\left(s_1, s_2, \ldots\right),$$

where B_n are Bell polynomials and

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(recall that only $\beta^{4k+1} \neq 0$ due to cyclicity of trace and symmetry of $\mathbb{\Lambda}$).

- Hence $\phi_G \wedge \phi_G = 0$, and therefore $\alpha_G \wedge \alpha_G = 0$.

Dipole sums in TQFT

- Recall $$\{\mathcal{O}_1, \mathcal{O}_2, \ldots\} = \sum_{\text{Graphs } G} \frac{1}{|\mathrm{Aut}(G)|} I_G \prod_{v \in V_G} \prod_i \varphi_{i,v}.$$

  symmetry factor

  Feynman integral

  External leg structure

- α_G is of form degree ℓ, so $\int_G \alpha_G \neq 0$ only if $|E| = \ell + 1$, which are multi-edge graphs (=dipoles) D_{2i+1} with $\ell = 2i$. Then $\alpha_{D_{2i+1}} \propto \Omega_{\ell+1}/\psi_G^{i+\frac{1}{2}}$ and $I_G = \int_G \alpha = \frac{1}{2^\ell}$.

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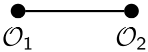
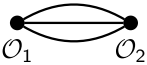
- For local operators $\mathcal{O}_1, \mathcal{O}_2$ (polynomials in p and q), propagator connects p with q,



$$= (\partial_p \mathcal{O}_1)\,(\partial_q \mathcal{O}_2) - (\partial_q \mathcal{O}_1)\,(\partial_p \mathcal{O}_2) =: \eta^{ij}(\partial_i \mathcal{O}_1)(\partial_j \mathcal{O}_2), \quad \eta := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$



$$= \quad (\partial_p^3 \mathcal{O}_1)\,(\partial_q^3 \mathcal{O}_2) - (\partial_q^2 \partial_p \mathcal{O}_1)\,(\partial_p^2 \partial_q \mathcal{O}_2) \pm \dots = \eta^{ij}\eta^{kl}\eta^{mn}(\partial_i \partial_k \partial_m \mathcal{O}_1)(\partial_j \partial_l \partial_n \mathcal{O}_2).$$

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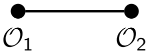

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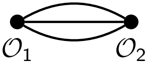

$$= (\partial_p^3 \mathcal{O}_1)(\partial_q^3 \mathcal{O}_2) - (\partial_q^2\partial_p \mathcal{O}_1)(\partial_p^2\partial_q \mathcal{O}_2) \pm \dots = \eta^{ij}\eta^{kl}\eta^{mn}(\partial_i\partial_k\partial_m \mathcal{O}_1)(\partial_j\partial_l\partial_n \mathcal{O}_2).$$

- The sum becomes the *Moyal commutator* $\{\mathcal{O}_1, \mathcal{O}_2\} = \mathcal{O}_1 \star \mathcal{O}_2 - \mathcal{O}_2 \star \mathcal{O}_1 = [\mathcal{O}_1, \mathcal{O}_2]_\star$,

$$\{\mathcal{O}_1, \mathcal{O}_2\} = \sum_{n=0}^{\infty} \frac{1}{4^n} \frac{1}{(2n+1)!} \left(\eta^{ij}\right)^{2n+1} \left(\partial^{2n+1}\mathcal{O}_1\right)\left(\partial^{2n+1}\mathcal{O}_2\right).$$

- Subtract anomaly to obtain a quantum corrected differential $Q' = Q - \{\cdot, \mathcal{O}\}$.

Dipole sums in graph complexes

- Dual complex of GC_3 has codifferential δ which acts by splitting a vertex, i.e. inserting an edge (=0-loop dipole D_0) into a vertex. Can be expressed as Lie bracket $\delta G \equiv [G, D_0] = G \circ D_0 - D_0 \circ G$.

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- The integral $\int_G \phi_G$ is non-zero iff G is a (linear combination of) dipoles. Hence, the Pfaffian can be viewed as a pairing with dipoles,

$$\int_G \phi_G = \langle G, \mathfrak{m} \rangle \qquad \text{with the dipole sum} \qquad \mathfrak{m} := \sum_{i=1}^{\infty} \frac{D_{2i+1}}{2(2i+1)!}.$$

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- Curious fact: The Lie bracket with $\mathfrak{m}$+(edge) is a codifferential, too [Khoroshkin, Willwacher, and Živković 2017], *twisted differential*

$$\delta' = \delta + [\cdot, \mathfrak{m}]\,.$$

(i.e. insert dipole sum instead of just one edge)
The cohomology of GC_3 wrt *this* codifferential δ' (instead of the usual δ) is 1-dimensional, with the only class is a sum of dipoles itself.

Stokes relations

- For Pfaffian-only $I_G = \int_G \phi_G$ (i.e. not wedged with canonical forms ω), have the Stokes relation [Brown, Hu, and Panzer 2024]

$$0 = I_{\partial G} + \frac{1}{2} \left[I_G , I_G \right].$$

- Choose G=triangle with dipole sides, then ∂G= dipole and one obtains recurrence

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- On the other hand, brackets $\{\cdot\}$ form L_∞ structure. Amounts to quadratic identities

$$\sum_{S \subset G,\ |V_S|=2} \operatorname{sgn}(G,S) \Delta_{G[S]} \times \Delta_{G/S} = 0$$

for their integration domain Δ (the operatope) [Gaiotto, Kulp, and Wu 2024; Budzik et al. 2023]. These are equivalent to the Stokes relations above.

Conclusion

- There is a certain, "topological", differential form α_G of degree ℓ in Schwinger parameters which computes BRST anomalies in TQFTs.
- There is another, "Pfaffian", differential form, ϕ_G, of degree ℓ which realizes the combinatorial sign of the odd graph complex GC_3 and therefore makes integrals $\int_G \phi_G \wedge \omega_G$ well-defined, where ω_G is a canonical form (which on its own lives on the even graph complex GC_2).
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- The two forms are the same.
- This leads to some simplified proofs for α_G, and to a physical interpretation for ϕ_G.
- The sum of dipole/multi-edge graphs plays a special role on both sides.
- Stokes relations have been known, and are important, on both sides.
- On both sides, one is interested in products between this form and some other forms.

Further investigations are currently in progress (with Simone Hu).

Thank you!

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The full Stokes relation

- Let $I_G(\omega) := \int_G \phi_G \wedge \omega_G$ for a canonical form ω which is not necessarily primitive $\Delta\omega = \mathbb{1} \otimes \omega + \omega \otimes \mathbb{1} + \omega' \otimes \omega''$ (i.e. consists of multiple factors β).
- Recall duality $(\delta I)_G \equiv I_{\partial G}$.
- Multi edge sum $\mathfrak{m}$.
- Lie bracket $[A, B] = A \circ B - B \circ A$ is mutual insertion.
- Stokes relation is

$$0 = \int_G \mathrm{d}\left(\phi \wedge \omega\right) = \delta I(\omega) + [I(\omega), \mathfrak{m}] + \frac{1}{2} \sum (-1)^{\left|\omega'\right|} \left[I(\omega''), I(\omega)\right].$$

Dodgson polynomials

- Consider the *expanded Laplacian*, defined as the block matrix

$$\mathbb{M} := \begin{pmatrix} \mathbb{D} & \mathbb{I} \\ -\mathbb{I}^T & 0 \end{pmatrix}.$$

  One can show that $\det(\mathbb{M}) = \psi$.

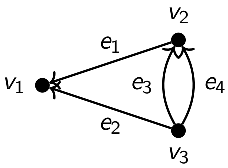
- Let $\mathbb{M}(A,B)$ be $\mathbb{M}$ with rows A and columns B removed. If $|A| = |B|$, this is a square matrix, and its determinant is called *Dodgson polynomial*

$$\psi^{A,B} := \det\big(\mathbb{M}(A,B)\big).$$

- In particular, if $A = \{i\}$ and $B = \{j\}$ each consist of only one index, the Dodgson polynomials $\psi^{i,j}$ are the cofactors of $\mathbb{M}$, i.e. they are entries of the inverse.
- $\mathbb{M}$ has block form, so $\mathbb{M}^{-1}$ has block form. Bottom right block is $\mathbb{L}^{-1}$. $\Rightarrow$ Lemma:

$$\left(\mathbb{L}^{-1}\right)_{i,j} = (-1)^{i+j}\frac{\psi^{i,j}}{\psi} \quad \text{(where i,j are indices of vertices).}$$

Example: Dogson polynomials



$$\mathbb{I} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \qquad \mathbb{M} = \begin{pmatrix} a_1 & 0 & 0 & 0 & 1 & -1 \\ 0 & a_2 & 0 & 0 & 1 & 0 \\ 0 & 0 & a_3 & 0 & 0 & 1 \\ 0 & 0 & 0 & a_4 & 0 & 1 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 & 0 \end{pmatrix}$$

In $\mathbb{M}$, the first 4 rows and columns refer to edges, the last 2 rows and columns refer to vertices v_1, v_2.

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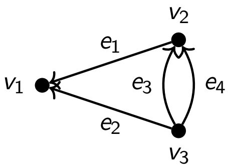

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In $\mathbb{M}$, the first 4 rows and columns refer to edges, the last 2 rows and columns refer to vertices v_1, v_2. Compute vertex-indexed Dodgson polynomials explicitly:

$$\psi^{v_1,v_1} = \det \begin{pmatrix} a_1 & 0 & 0 & 0 & -1 \\ 0 & a_2 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 1 \\ 0 & 0 & 0 & a_4 & 1 \\ 1 & 0 & -1 & -1 & 0 \end{pmatrix} = a_2 \left(a_1 a_3 + a_1 a_4 + a_3 a_4 \right)$$

$$\psi^{v_1,v_2} = -a_2 a_3 a_4 = \psi^{v_2,v_1}, \qquad \psi^{v_2,v_2} = (a_1 + a_2) a_3 a_4.$$

Indeed,

$$\mathbb{L}^{-1} = \frac{1}{a_3 a_4 + a_1 a_3 + a_1 a_4 + a_2 a_3 + a_2 a_4} \begin{pmatrix} a_2 (a_3 a_4 + a_1 (a_3 + a_4)) & a_2 a_3 a_4 \\ a_2 a_3 a_4 & (a_1 + a_2) a_3 a_4 \end{pmatrix}.$$

Background: Deformation quantisation

- Given is a *classical field theory*: Smooth manifold M. Field variable $\phi(t, \mathbf{x})$, canonical conjugate $\pi(t, \mathbf{x})$ are smooth functions on M. Hamilton function $H(\phi(t, \mathbf{x}), \pi(t, \mathbf{x}))$. Skew-symmetric *Poisson bracket* $\{f, g\} \in C^\infty(M)$. Gives equations of motion:

$$\partial_t \phi = \{\phi, H\}, \qquad \partial_t \pi = \{\pi, H\}, \qquad \{\phi, \pi\} = 1.$$

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- Naive quantisation: Replace $\{f, g\}$ by $\frac{i}{\hbar}\left[\hat{f}, \hat{g}\right]$. Runs into inconsistencies for powers of fields. Deformation quantisation: Find a "star product" $\star$ such that

$$[f, g]_\star := f \star g - g \star f \overset{!}{=} \hbar \{f, g\} + \mathcal{O}\left(\hbar^2\right).$$

- Power series ansatz with (to be determined) differential operators $B_j(f, g)$.

$$f \star g = B_0(f, g) + \hbar B_1(f, g) + \hbar^2 B_2(f, g) + \ldots,$$

  Clearly $B_0(f, g) = f \cdot g$ and $B_1(f, g) = \frac{1}{2}\{f, g\}$. What are the higher B_j?
- Two conditions:
  1. Should be associative $f \star (g \star h) = (f \star g) \star h$,
  2. Should be invariant under diffeomorphisms $f \mapsto f + \hbar D_1(f) + \hbar^2 D_2(f) + \ldots$.

Background: Deformation quantisation 2

- Solution in [Kontsevich 2003]: Consider graphs Γ embedded in the upper half plane $\mathbb{H} = \{z \in \mathbb{C} | \Im(z) > 0\}$ with hyperbolic metric.
- In Γ, each vertex with 2 outgoing edges corresponds to a factor $\omega^{ij}\partial_i\partial_j$. (i.e. a graph Γ encodes a nesting of Poisson brackets, a differential operator B_Γ). Graph has n upper vertices and 2 vertices at bottom line $\mathbb{R}$, corresponding to arguments f, g of $B_n(f, g)$.

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- Define angle $\phi(p, q)$ between geodesic $p \longrightarrow q$ and vertical line $p \longrightarrow i\infty$.
- Each graph is weighted by a weight integral $W_\Gamma = \text{const} \times \int \bigwedge_{e \in E_\Gamma} \mathrm{d}\phi_e$. Star product is (details omitted)

$$\star = \cdot \ + \sum_{n=1}^{\infty} \hbar^n \sum_{\Gamma} W_\Gamma B_\Gamma .$$

Background: Deformation quantisation 3

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- Crucial step: Show that the so-defined $\star$ is associative.
- Associativity condition at order $\hbar^n$,

$$\sum_{k=0}^{n} B_k\left(B_{n-k}(f,g),h\right)=\sum_{k=0}^{n} B_k\left(f,B_{n-k}(g,h)\right),$$

  amounts to insertion of operators B_j, hence nesting/shrinking of graphs.
- Obstructions to associativity are given by certain integrals over the boundary of configuration space,

$$c_\Gamma=\int\limits_{\partial\bar{C}_{n,m}}\bigwedge_{e\in E_\Gamma}\mathrm{d}\phi_e.$$

  These integrals can be shown to vanish and $\star$ is associative. More general, abstract statement: "Formality theorem".

The significance of $\alpha_\Gamma \wedge \alpha_\Gamma$

- α_Γ is a differential form in $\mathrm{d}a_e$, to be integrated over Schwinger parameters a_e. Itself, α_Γ is an integral over vertex positions x_v of some integrand W_Γ. Schematically:

$$\mathcal{F}(\Gamma) = \int_{\{a_e\}} \alpha_\Gamma = \int_{\{a_e\}} \int_{\{x_v\}} W_\Gamma, \qquad W_\Gamma = \bigwedge_{e \in E_\Gamma} e^{-s_e^2}\, \mathrm{d}s_e.$$

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- There is one Schwinger variable a_e for each edge, but there could be more than one coordinate x_v for each vertex (i.e. the vertex coordinate is a vector $(x_v^{(1)}, x_v^{(2)}, \ldots)$. Consider a 2-dimensional theory

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- Here, $\alpha_\Gamma \wedge \alpha_\Gamma$ is some differential form in the $\mathrm{d}a_e$'s, independent of the x_v. Conversely, we can exchange the order of integration and do the $\mathrm{d}a_e$ integral first. The integrand is

$$W_\Gamma^{(1)} \wedge W_\Gamma^{(2)} = \exp\left(-\sum_e \left({s_e^{(1)}}^2 + {s_e^{(2)}}^2 \right) \right) \bigwedge_e \mathrm{d}s_e^{(1)} \wedge \mathrm{d}s_e^{(2)} .$$

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$$\int_{\{a_e\}} W_\Gamma^{(1)} \wedge W_\Gamma^{(2)} = \int_{\{a_e\}} \exp\left(-\sum_e \left({s_e^{(1)}}^2 + {s_e^{(2)}}^2\right)\right) \bigwedge_e \mathrm{d}s_e^{(1)} \wedge \mathrm{d}s_e^{(2)}$$

- This expression factorizes for edges. Consider an edge e from point $(0,0)$ to $(x^{(1)}, x^{(2)})$:

$$e^{-{s_e^{(1)}}^2 - {s_e^{(2)}}^2}\, \mathrm{d}s_e^{(1)} \wedge \mathrm{d}s_e^{(2)} = e^{-\frac{\vec{x}^2}{a}} \left(-2a_e^{-2}\,\mathrm{d}a_e \left(x^{(2)}\,\mathrm{d}x^{(1)} - x^{(1)}\,\mathrm{d}x^{(2)}\right) + a_e^{-1}\,\mathrm{d}x^{(1)} \wedge \mathrm{d}x^{(2)}\right).$$

The significance of $\alpha_\Gamma \wedge \alpha_\Gamma$

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- Only the term $\propto \mathrm{d}a_e$ contributes to integral. Polar coordinates in the plane:

  $$\vec{x} = \begin{pmatrix} x^{(1)} \\ x^{(2)} \end{pmatrix} = r \begin{pmatrix} \sin\phi \\ -\cos\phi \end{pmatrix}, \qquad \frac{\mathrm{d}\vec{x}}{\mathrm{d}\phi} = r \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} = \begin{pmatrix} -x^{(2)} \\ x^{(1)} \end{pmatrix}.$$

  $\Rightarrow \qquad x^{(1)}\, \mathrm{d}x^{(2)} - x^{(2)}\, \mathrm{d}x^{(1)} = \left((-x^{(2)})^2 + (x^{(1)})^2 \right) \mathrm{d}\phi = |\vec{x}|^2\, \mathrm{d}\phi$ is the differential of the 2D angle ϕ of the vector $\vec{x}$.

The significance of $\alpha_\Gamma \wedge \alpha_\Gamma$

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- Integrate the Schwinger parameter a_e for a single edge:

$$\int_{a_e=0}^{\infty} e^{-s^{(1)2} - s^{(2)2}}\, \mathrm{d}s^{(1)} \wedge \mathrm{d}s^{(2)} = \int_{a_e=0}^{\infty} e^{-\frac{|\vec{x}|^2}{a_e}} 2a_e^{-2}\, |\vec{x}|^2\, \mathrm{d}\phi_e \wedge \mathrm{d}a_e = 2\, \mathrm{d}\phi_e.$$

The significance of $\alpha_\Gamma \wedge \alpha_\Gamma$

- We conclude that the 2-dimensional integral is (very schematically)

$$\mathcal{F}(\Gamma) = \int_{\left\{x_v^{(1)}\right\}} \int_{\left\{x_v^{(2)}\right\}} \int_{\{a_e\}} W_\Gamma^{(1)} \wedge W_\Gamma^{(2)} = \int_{\{\text{relative positions } \vec{x}_v\}} \bigwedge_e \mathrm{d}\phi_e.$$

Closer investigation of the last integral shows: These are the Kontsevich integrals c_Γ which need to vanish in order to make the star product associative and establish the *formality theorem*.

The significance of $\alpha_\Gamma \wedge \alpha_\Gamma$

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  Closer investigation of the last integral shows: These are the Kontsevich integrals c_Γ which need to vanish in order to make the star product associative and establish the *formality theorem*.

- On the other hand:

$$c_\Gamma = \mathcal{F}(\Gamma) = \int_{\{a_e\}} \int_{\left\{x_v^{(1)}\right\}} \int_{\left\{x_v^{(2)}\right\}} W_\Gamma^{(1)} \wedge W_\Gamma^{(2)} = \int_{\{a_e\}} \alpha_\Gamma \wedge \alpha_\Gamma$$

- Hence $\int_{\{a_e\}} \alpha_\Gamma \wedge \alpha_\Gamma = 0$ implies the vanishing of Kontsevich integrals.

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