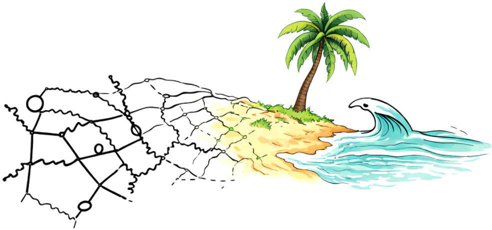
MA∫CAMP

31 March 2026, ECT*, Trento

Non-perturbative renormalization in fundamental and effective theories

University of Oxford Mathematical Institute

Tropical renormalization at 400 loops



Paul-Hermann Balduf

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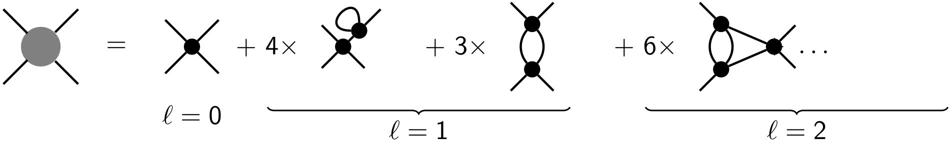
Based on math-ph/2512.21091 with Erik Panzer (the arXiv version will be updated next week).

Slides, data sets etc. gradually become available from paulbalduf.com/research.

Did you know about Mastodon, the friendly open source social network? **math**stodon.xyz/@paulbalduf

Reminder: Perturbative QFT

- We work in Euclidean quantum field theory, use ϕ^4_D as example, where $D = 4 - 2\epsilon$.
- Perturbation theory in terms of Feynman integrals; Feynman diagrams with vertex $(-\lambda)$, propagator $1/(p^2 + m^2)$. Graded by *loop number* ℓ of the diagram.



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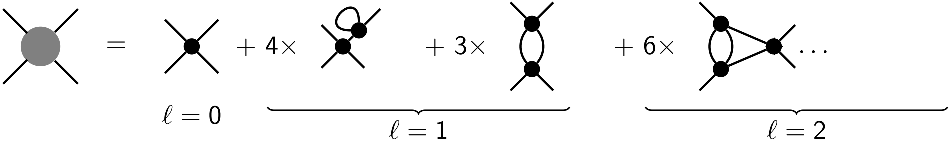

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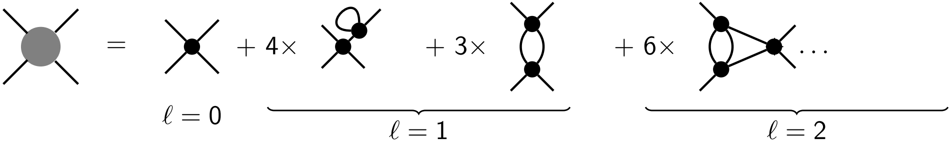

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  2. The number of diagrams, and hence integrals, grows factorially with ℓ [Bessis, Itzykson, and Zuber 1980]. The renormalized series is divergent and needs resummation [Lipatov 1977; Parisi 1978]. Encodes non-perturbative physics (first mentioned in [Dyson 1952], modern broader framework: *resurgence* [Ecalle 2022]).

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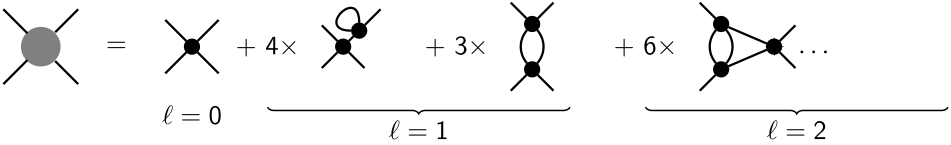

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- Tropical field theory is useful for either point:
  It simplifies numerical computation of Feynman integrals (not discussed today),
  and it is a model theory to study large-order asymptotics (discussed today at the end).

What is tropical field theory?

I will mention three different perspectives: (see above)

1. An approximation scheme for Feynman integrals,
2. A limit of long-range field theory,
3. A (generalization of) a certain model in functional renormalization group theory.

Perspective 1: Parametric Feynman integrals

- Leave out vertex Feynman rules $(-\lambda)$. Schwinger-parametric representation of propagator:

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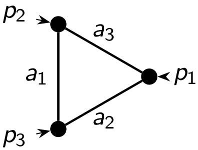
- Replace all propagators in the diagram G, impose momentum conservation, obtain Gaussian integral in p^2, solve. Find *parametric Feynman integral* [Nakanishi 1957]:

$$\mathcal{I}[G] = \Gamma(\omega_G) \int_0^\infty \mathrm{d}a_1 \cdots \int_0^\infty \mathrm{d}a_{|E|-1}\ \frac{1}{\mathcal{U}^{\frac{D}{2}}} \left(\frac{\mathcal{U}}{\mathcal{F}}\right)^{\omega_G} \Bigg|_{a_{|E|}=1}.$$

- The superficial degree of convergence is $\omega_G = |E_G| - \ell \frac{D}{2}$.
Symanzik polynomials, where $p_F =$ sum of momenta entering F, are

$$\mathcal{U} := \sum_{T \text{ spanning tree}} \prod_{e \notin T} a_e, \qquad \mathcal{F} := \sum_{\text{F spanning 2-forest}} p_F^2 \prod_{e\notin F} a_e + \sum_e m^2 a_e \cdot \mathcal{U}.$$

Example: Triangle



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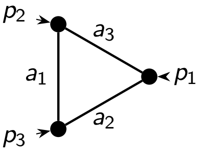

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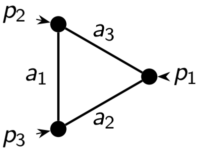

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- Parametric Feynman integral is 2-dimensional,

$$\mathcal{I}[G] = \Gamma(1+\epsilon) \int_0^\infty \mathrm{d}a_1 \int_0^\infty \mathrm{d}a_2 \, \frac{1}{\mathcal{U}^{2-\epsilon}} \left(\frac{\mathcal{U}}{\mathcal{F}} \right)^{1+\epsilon} \Bigg|_{a_3=1}.$$

Restricting to the static case

- Polynomial $\mathcal{F}$ depends on all momenta and masses[1].

[1] Compare discussion yesterday: kinematic singularities at values $(p_1, p_2, \ldots, m)$ such that $\mathcal{F} = 0$ and $\partial_a \mathcal{F} = 0$ [Landau 1959].

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- For this talk: Set all $p = 0$, and assume all masses are equal. Called the *static* case. (one can also set up tropical field theory with $p \neq 0$, but formulas are too long for the slides)

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- This implies that $\frac{\mathcal{F}}{\mathcal{U}} = m^2 \sum_e a_e$. The Feynman integral is then proportional to m:

$$\mathcal{I}[G] = \Gamma(\omega_G) \cdot m^{-2\omega_G} \cdot \int_0^\infty \mathrm{d}a_1 \cdots \int_0^\infty \mathrm{d}a_{|E|-1} \left. \frac{\left(\sum_e a_e\right)^{-\omega_G}}{\mathcal{U}^{\frac{D}{2}}} \right|_{a_{|E|}=1} .$$

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Triangle at zero momentum

- $\mathcal{U} = a_1 + a_2 + a_3$ and $\mathcal{F} = m^2 \sum_e a_e \mathcal{U}$, so $\frac{1}{\mathcal{U}^{2-\epsilon}} \left(\frac{\mathcal{U}}{\mathcal{F}} \right)^{1+\epsilon} = \frac{(m^2)^{-1-\epsilon}}{\mathcal{U}^{2-\epsilon}} \left(\sum_e a_e \right)^{-1-\epsilon}$.

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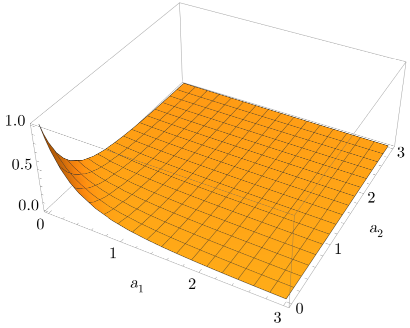
- Feynman integral becomes

$$\mathcal{I}[G] = (m^2)^{-1-\epsilon}\Gamma(1+\epsilon)\int_0^\infty \mathrm{d}a_1 \int_0^\infty \mathrm{d}a_2 \, \frac{1}{(a_1+a_2+1)^3} = (m^2)^{-1-\epsilon}\Gamma(1+\epsilon)\cdot \frac{1}{2}.$$

- Plot the integrand $\frac{1}{(a_1+a_2+1)^3}$:



Tropical approximation

- Goal: Construct an approximation for the integral

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- We integrate over $a_e \geq 0$, and all coefficients in $\mathcal{U}$ are positive. $\Rightarrow$ for fixed $a_1, \ldots, a_{|E|-1}$,

$$\mathcal{U}(a) = \sum \left(\text{positive monomials} \prod_e a_e\right).$$

Hence $\mathcal{U} \geq$ (largest monomial), or equivalently $\frac{1}{\mathcal{U}} \leq \frac{1}{(\text{largest monomial})}$.

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- Concretely, for given numerical values of $a_1, \ldots, a_{|E|-1}$, replace

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- This operation is mathematically a *tropicalization* of a polynomial [Speyer and Sturmfels 2009] (named after Christian Choffrut and Imre Simon working in São Paulo, Brasil [Simon 1988]).
- Likewise, replace $\sum_e a_e \to \max_e a_e$. The integrand is then piecewise monomial.

Tropical integral of the triangle

- Integrand is $\frac{1}{\mathcal{U}^3}$, replace by $\frac{1}{{\mathcal{U}_{\mathrm{tr}}}^3}$,

$$\mathcal{U}_{\mathrm{tr}} = \max\{a_1, a_2, 1\} = \begin{cases} a_1 & \text{if } a_1 > 1 \ \& \ a_1 > a_2, \\ a_2 & \text{if } a_2 > 1 \ \& \ a_2 > a_1, \\ 1 & \text{else.} \end{cases}$$

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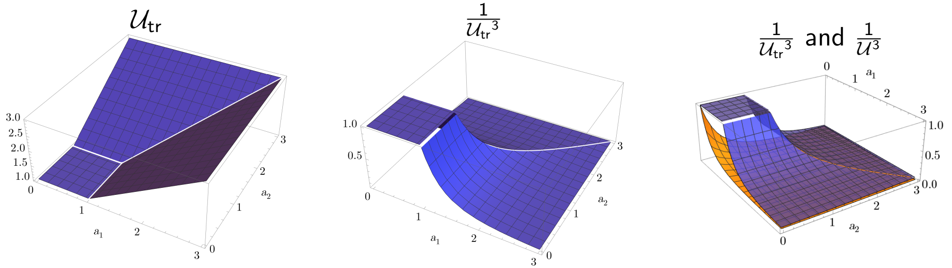

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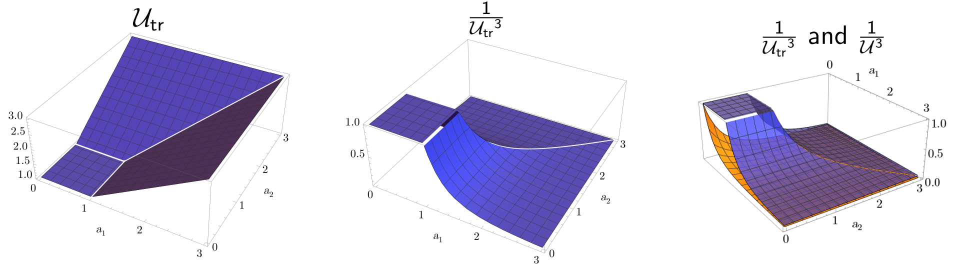

- Solving the integral over $\frac{1}{{\mathcal{U}_{\text{tr}}}^3}$ is easy when splitting the domain of integration:

$$\int_0^1 \mathrm{d}a_1 \int_0^1 \mathrm{d}a_2 \frac{1}{1^3} + \int_1^\infty \mathrm{d}a_1 \int_0^{a_1} \mathrm{d}a_2 \frac{1}{a_1^3} + \int_1^\infty \mathrm{d}a_2 \int_0^{a_2} \mathrm{d}a_1 \frac{1}{a_2^3} = 1 + 1 + 1 = 3.$$

Structural features of the tropical approximation

- The tropicalization captures the true behaviour as $a_j \to \infty$ (i.e. UV divergences including subdivergences). Modulo constants, $\mathcal{I}_{\mathrm{tr}}[G]$ is a strict upper and lower bound ("Hepp bound") of $\mathcal{I}[G]$.
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- Hepp sectors correspond to spanning trees. An integral over a monomial is trivial; solving the tropical Feynman integral is combinatorics.
- Convention: replace the leading $\Gamma(\omega_G)$ by $\frac{1}{\omega_G}$. In the static limit, the tropical integral $\mathcal{I}_{\text{tr}}[G]$ is a rational function of ϵ, general combinatorial formula [Panzer 2022]:

$$\mathcal{I}_{\text{tr}}[G] = m^{-2\omega_G} \sum_{\sigma} \frac{1}{\omega_{G_1^\sigma} \omega_{G_2^\sigma} \cdots \omega_{G_{|E|-1}^\sigma} \omega_{G_{|E|}^\sigma}} = m^{-2\omega_G} \frac{1}{\omega_G} \sum_{e \in E_G} \mathcal{I}_{\text{tr}}[G \setminus e],$$

where σ is a permutation of edges, and G_k^σ is the graph built from the first k edges of σ.

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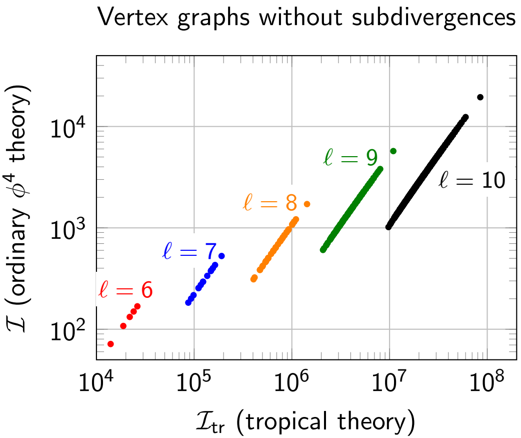

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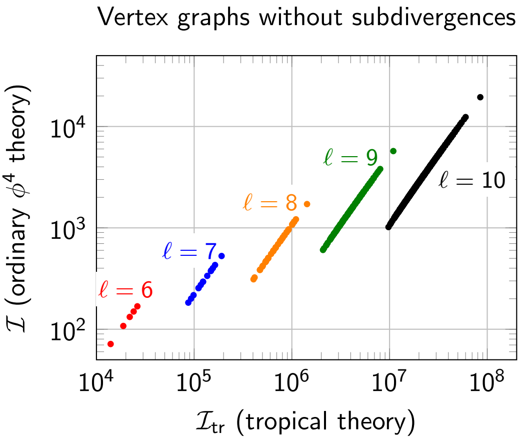
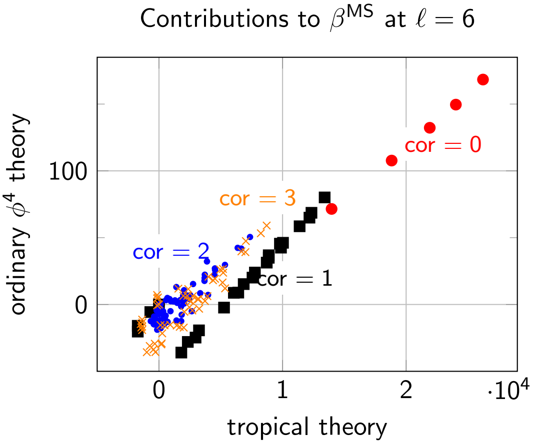
- With subdivergence, consider contribution to β^{MS} (i.e. $\frac{1}{\epsilon}$ residue of counterterm).





Perspective 2: Tropicalization as an analytic limit

- Have discussed ad-hoc approximation given by *tropicalization*

$$\mathcal{U}(a) = \sum_{T \text{ spanning tree}} \prod_{e \notin T} a_e \quad \to \quad \max_{T \text{ spanning tree}} \prod_{e \notin T} a_e =: \mathcal{U}_{\text{tr}}(a).$$

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- Recall L_∞-norm from calculus lecture:

$$\lim_{\xi \to 0} \left(c_1 y_1^{\frac{1}{\xi}} + c_2 y_2^{\frac{1}{\xi}} + \ldots \right)^{\xi} = \max\{y_1, y_2, \ldots\} \qquad \text{(the } c_j \neq 0 \text{ don't matter).}$$

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- We get such exponents in $\mathcal{U}$ if we use a propagator with non-unit power,

$$\frac{1}{(p^2 + m^2)^{\xi}} = \frac{1}{\Gamma(\xi)} \int_0^\infty \mathrm{d}a \, a^{\xi - 1} e^{-a(p^2 + m^2)} = \frac{1}{\xi \Gamma(\xi)} \int_0^\infty \mathrm{d}y \, e^{-y^{\frac{1}{\xi}} (p^2 - m^2)}.$$

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- Integrand is $\mathcal{U}^{-\frac{D}{2}}$. To get overall power ξ in the exponent, need $D = \xi \cdot (4-2\epsilon)$.
- Introduce $\bar{\omega}_G = \xi \cdot \omega_G$ (where $\omega_G = |E| - \ell \frac{4-2\epsilon}{2}$). Then $\lim_{\xi \to 0} \mathcal{I} = \mathcal{I}_{\text{tr}}$.

Long-range ϕ^4 theory

- Long-range ϕ^4 theory (e.g. [Dauxois et al. 2002; Benedetti et al. 2020], many more)

$$\mathcal{L} = \frac{1}{2}\phi\left(-\partial_\mu \partial^\mu + \bar{\mu}^2\right)^\xi \phi + \frac{1}{2} m^{2\xi}\phi^2 + \frac{(4\pi)^2 \lambda}{4!}\phi^4.$$

- $\bar{\mu}$ is an IR regulator and m the particle mass (coincide for short-range theory $\xi = 1$). The long-range propagator $\frac{1}{(p^2+\bar{\mu}^2)^\xi}$ represents a *massless* long-range theory. We treat m as a perturbation (=2-valent vertex).
- Mass dimensions $[\phi] = \frac{D-2\xi}{2}$ and $[\mu] = [m] = 1$ and $[\int \mathrm{d}^D x \phi^4] = D - 4\xi$.

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- $\bar{\mu}$ is an IR regulator and m the particle mass (coincide for short-range theory $\xi = 1$). The long-range propagator $\frac{1}{(p^2+\bar{\mu}^2)^\xi}$ represents a *massless* long-range theory. We treat m as a perturbation (=2-valent vertex).
- Mass dimensions $[\phi] = \frac{D-2\xi}{2}$ and $[\mu] = [m] = 1$ and $[\int \mathrm{d}^D x\phi^4] = D - 4\xi$.
- We impose $D = \xi(4 - 2\epsilon)$, and rescale $\mu := \bar{\mu}^\xi$. Then all Feynman integrals are proportional to $\mu^{-2\omega_G}$. This is perspective 2 on (static) tropical field theory:

Tropical field theory is the limit $\xi \to 0$ of this long-range ϕ^4 theory.

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Tropical field theory is the limit $\xi \to 0$ of this long-range ϕ^4 theory.

- This is a "blow-up" of a singular limit, i.e. the parameter ϵ encodes "direction" when taking $\xi \to 0$. One can not simply set $D = 0$ or $\xi = 0$ individually.

Perspective 3: Based on combinatorial formula for tropical integrals

- Recall combinatorial formula for (static) tropical integral,

$$\mathcal{I}_{\text{tr}}[G] = \mu^{-2\omega_G} \sum_{\sigma} \frac{1}{\omega_{G_1^\sigma}\omega_{G_2^\sigma} \cdots \omega_{G_{|E|-1}^\sigma}\omega_{G_{|E|}^\sigma}} = \mu^{-2\omega_G} \frac{1}{\omega_G} \sum_{e \in E_G} \mathcal{I}_{\text{tr}}[G \setminus e].$$

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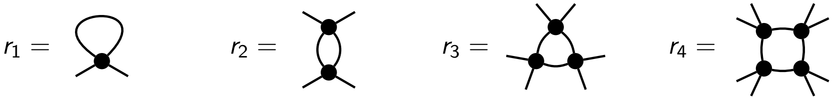
- Trees have $\mathcal{I}_{\text{tr}}[T] = 1$.
- For example, 1-loop cycles r_n with n edges. It has $\omega_{r_n} = n - 2 + \epsilon$.

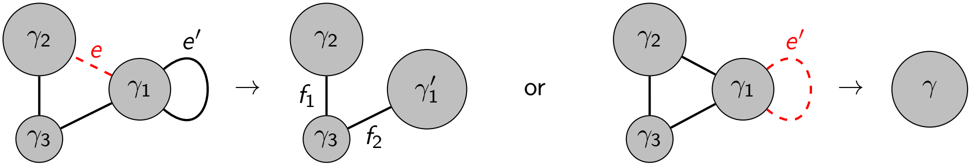


- Removing any one edge produces a tree. Since $\mathcal{I}_{\text{tr}}[T] = 1$ for every tree, we have

$$\mathcal{I}_{\text{tr}}[r_n] = \mu^{-2n+4-2\epsilon} \frac{1}{\omega_G} \sum_{e \in E_G} \mathcal{I}_{\text{tr}}[G \setminus e] = \mu^{-2n+4-2\epsilon} \frac{1}{\epsilon + n - 2} \sum_{e \in E_G} 1 = \mu^{-2n+4-2\epsilon} \frac{n}{\epsilon + n - 2}.$$

Recurrence relations from graphs

- Remove 1 edge from arbitrary 1PI graph. Produce either a tree of 1PI graphs, or not:



Removing 1 edge reduces loop number by 1. $\Rightarrow$ recurrence relation for 1PI graphs.

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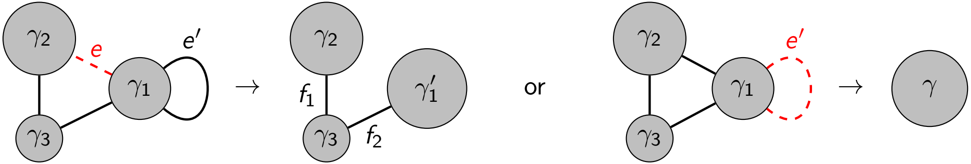

Removing 1 edge reduces loop number by 1. ⇒ recurrence relation for 1PI graphs.

- Introduce (unrenormalized) 1PI generating function

$$\mathcal{G}(\varphi,\lambda,\kappa) := \sum_{n=0}^{\infty}\sum_{t=0}^{\infty}\sum_{s=0}^{\infty}\Gamma^{(t)}_{n,s}\kappa^s\lambda^t\frac{\varphi^n}{n!} = -\frac{\varphi^2}{2} - \left(\frac{N(N+2)}{24(\epsilon-1)^2} + \frac{(N+2)\varphi^2}{12(\epsilon-1)} + \frac{\varphi^4}{24}\right)\lambda + \mathcal{O}(\kappa) + \ldots .$$

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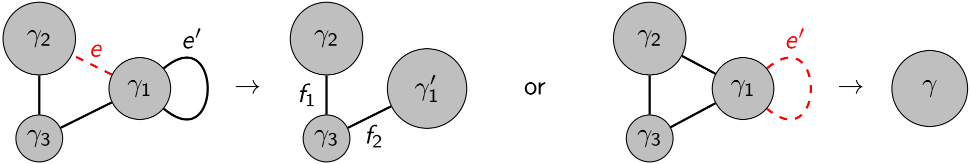

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- Satisfies the *tropical loop equation* [Borinsky 2025]

$$\Big(2\epsilon\lambda\partial_\lambda + (1-\epsilon)\varphi\partial_\varphi + 2\kappa\partial_\kappa - (4-2\epsilon)\Big)\mathcal{G} = \varphi^2 - \frac{1}{\partial_\varphi^2\mathcal{G}} - \frac{(N-1)\varphi}{\partial_\varphi\mathcal{G}} - N.$$

Divergence structure

- When does the tropical Feynman integral have poles in ϵ?

$$\mathcal{I}_{\text{tr}}[G] = \mu^{-2\omega_G} \sum_{\sigma} \frac{1}{\omega_{G_1^\sigma} \omega_{G_2^\sigma} \cdots \omega_{G_{|E|-1}^\sigma} \omega_{G_{|E|}^\sigma}}.$$

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- Inserting 2-point vertex κ produces log-divergence, the **mass term is renormalized**, the IR regulator/scale variable μ not.
- Multiplicative renormalization as usual; introduce reference scale μ_0 and counterterms:

$$\lambda = \mu_0^{2\epsilon} g \cdot Z_g(g,\epsilon), \qquad \kappa = m^2 \cdot Z_m(g,\epsilon).$$

Renormalized tropical loop equation

- Renormalized 1PI generating function $\mathcal{G}_{\mathcal{R}}\left(\varphi, g, m^2\right) = \mathcal{G}\left(\varphi, \mu^{2\epsilon} g Z_g, m^2 Z_m\right)$ satsifies the tropical loop equation

$$\Big(-\beta(g,\epsilon)\partial_g + (1-\epsilon)\varphi\partial_\varphi + \big(2+\gamma_m(g,\epsilon)\big)m^2\partial_{m^2} - (4-2\epsilon)\Big)\mathcal{G}_{\mathcal{R}} = \varphi^2 - \frac{1}{\partial_\varphi^2 \mathcal{G}_{\mathcal{R}}} - \frac{(N-1)\varphi}{\partial_\varphi \mathcal{G}_{\mathcal{R}}} - N,$$

where we identified the inner derivatives as the *renormalization group functions*

$$\beta(g,\epsilon) := \mu_0 \frac{\partial}{\partial \mu_0} g(\lambda, \mu_0, \epsilon), \qquad \gamma_m(g,\epsilon) := \mu_0 \frac{\partial}{\partial \mu_0} \ln m^2(\lambda, \mu_0, \epsilon).$$

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- Of course, the Callan-Symanzik equation holds, using $t := \log\left(\frac{\mu_0}{\mu}\right)$:

$$\left(\beta \cdot \partial_g - \gamma_m \cdot m^2 \partial_{m^2}\right) G_{\mathcal{R}} = -\partial_t G_{\mathcal{R}}.$$

Flow equations

- Legendre transform to connected generating function (Helmholtz free energy)

$$j := \partial_\varphi \mathcal{G}_\mathcal{R}, \qquad \mathcal{W} := j \cdot \varphi - \mathcal{G}_\mathcal{R}.$$

- Split off free field theory $\frac{j^2}{2}$ to obtain non-free part of free energy/partition function

$$\mathcal{W} =: -\frac{j^2}{2} + u(j,t,m^2), \qquad \rho(j,t,m^2) := e^{-u} = e^{-\frac{j^2}{2}} \mathcal{Z}.$$

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- With the scale variable $t = \log \frac{\mu_0}{\mu}$ and spacetime dimension D, the static limit of tropical ϕ^4 theory satisfies the renormalization flow equations

$$\partial_t u = \partial_j^2 u - \frac{D-2}{2} \cdot j \partial_j u + D \cdot u - (\partial_j u)^2 \; + \frac{N-1}{j} \partial_j u,$$
$$\partial_t \rho = \partial_j^2 \rho - \frac{D-2}{2} \cdot j \partial_j \rho + D \cdot \rho \log \rho \; + \frac{N-1}{j} \partial_j \rho.$$

These equations have been known for decades!

[Felder 1987]

Commun. Math. Phys. 111, 101–121 (1987)

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© Springer-Verlag 1987

Renormalization Group in the Local Potential Approximation

Giovanni Felder

Institut des Hautes Etudes Scientifiques, 35, route de Chartres, F-91440 Bures-sur-Yvette, France

In this paper, we study a continuous scale version of the hierarchical model. It is given by the partial differential equation

$$u_t = \tfrac{1}{2}u_{xx} - \frac{d-2}{2}xu_x + du - \tfrac{1}{2}u_x^2, \qquad (1.1)$$

describing the flow of the effective potential $u(t,x)$ on momentum scale e^{-t} as a function of the field $x \in \mathbb{R}$. A similar equation was studied numerically by Hasenfratz and Hasenfratz [9] who found a non-trivial fixed point in three dimensions. Brydges and Kennedy [10] also studied similar equations in connection with the Mayer expansion.

[Zumbach 1994]

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NUCLEAR PHYSICS B [FS]

The renormalization group in the local potential approximation and its applications to the O(n) model

Gil Zumbach

Harvard University, Department of Physics, Cambridge MA 02138, USA

Received 28 July 1993
(Revised 6 September 1993)
Accepted for publication 10 September 1993

enough so as to contain the main physics. In this paper, we want to explore these two issues on the simplest approximation of the RG called the local potential approximation (LPA). For a model with n (boson) fields $\varphi_i, i = 1, \dots, n$, the equation is

$$\partial_t \mu(\varphi_i, t) = \partial_i^2 \mu - \frac{d-2}{2}\varphi_i \partial_i \mu + d\mu \ln \mu, \qquad (1)$$

where $\mu(\varphi_i, t) = \exp(-v(\varphi_i, t))$ is the density corresponding to the potential v, $\partial_i = \partial/\partial\varphi_i$, d is the space dimensionality and the RG "time" t corresponds to the cut-off $t = -\ln(\Lambda/\Lambda_0)$. At this point, we make no assumption about the symmetries of μ with respect to φ_i.

⇒ static tropical ϕ^4 theory coincides with the local potential approximation of ϕ^4.

What is tropical field theory good for?

- Proof of equivalence of the three setups, mutual interpretation.
- Very powerful for numerical integration of Feynman integrals in non-tropical theories [Borinsky 2023; Borinsky, Munch, and Tellander 2023; Borinsky and Fraaije 2025; Borinsky 2025].
- Here: Use tropical QFT to study structural features of renormalization.

Tropical beta function

- Consider beta function in perturbation theory,

$$\beta(g,\epsilon) := \frac{-2\epsilon g}{1 + g\partial_g \ln Z_g(g,\epsilon)} \overset{\text{in MS}}{=} -2g\epsilon + 2g^2 \partial_g [\epsilon^{-1}] Z_g(g).$$

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- Beta function of tropical ϕ^4 theory in the MS scheme:

$$\beta^{(\text{MS})} = -2\epsilon g + 6g^2 - 36g^3 + 522g^4 - 11256g^5 + \frac{1224063}{4}g^6 - \frac{97292007}{10}g^7 \pm \ldots.$$

- For comparison, in non-tropical $\phi^4_{4-2\epsilon}$ theory [Schnetz 2023]:

$$\beta \approx -2\epsilon g + 3g^2 - 5.667g^3 + 32.55g^4 - 271.6g^5 + 2849g^6 - 34776g^7 + \ldots.$$

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- Alternatively, define a "kinematic"/MOM scheme by evaluating at $\mu = \mu_0$,

$$g = g(u,\epsilon) := G_4(g, \mu = \mu_0, \epsilon), \qquad m = m(g,\epsilon) := \kappa G_{2,1}(u, \mu = \mu_0, \epsilon).$$

- Viewed as counterterm, this is simply $Z_g(g,\epsilon) = 1/G_4(\mu = \mu_0)$. Leads to

$$\beta^{(\text{MOM})} = -2\epsilon g + 6g^2 - \frac{18(3\epsilon - 2)}{(\epsilon+1)(\epsilon-1)}g^3 + \frac{18(-49 + 196\epsilon - 164\epsilon^2 - 94\epsilon^3 + 120\epsilon^4)}{(\epsilon-1)^2(\epsilon+1)(\epsilon+2)(2\epsilon-1)(2\epsilon+1)}g^4 + \ldots$$

$$= 6g^2 - 36g^3 + 441g^4 - 8061g^5 + \frac{735039}{4}g^6 - \frac{39225231}{8}g^7 \pm \ldots + \mathcal{O}(\epsilon).$$

Factorial growth rates and Borel transform

- Perturbation series are factorially divergent. Qualitatively for $n \to \infty$:

$$[g^n]\beta(g) = \beta_n \sim S \cdot \Gamma(n+b) \cdot a^n + \text{subleading corrections},$$

  with constants S, b, a.
- The *Borel transform* of β has non-zero radius of convergence;

$$\mathcal{B}[\beta](u) := \sum_{t=0}^{\infty} \frac{\beta_n}{n!} u^n.$$

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$$\beta_n = \frac{S}{\Gamma(b)} \cdot \Gamma(n+b) \cdot \left(\frac{1}{u_0}\right)^n \quad \Leftrightarrow \quad \mathcal{B}[\beta](u) \quad = \frac{S}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(n+b)}{\Gamma(n+1)} \left(\frac{u}{u_0}\right)^n = S \cdot \left(1 - \frac{u}{u_0}\right)^{-b}.$$

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- Borel resummation is given by the Laplace-Borel integral

$$\mathfrak{R}[\beta](g) = \frac{1}{g} \int_0^\infty \mathrm{d}u \, e^{-\frac{u}{g}} \mathcal{B}[\beta](t).$$

$\Rightarrow$ For physics, behaviour of $\mathcal{B}[\beta](u)$ along positive real line in Borel plane is crucial.

Literature expectations for Borel plane of β

[Beneke 1999]

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Physics Reports 317 (1999) 1–142

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www.elsevier.com/locate/physrep

Renormalons

M. Beneke

2.4. The Borel plane

We summarize what is known about singularities in the Borel plane. Recalling the definition of the Borel transform (2.5), the Borel plane for the Adler function (current–current correlation functions) is portrayed in Fig. 3. Note that the figure does not show what is *not* known. We

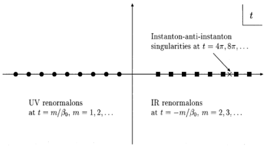


Fig. 3. Singularities in the Borel plane of $\Pi(Q^2)$, the current–current correlation function in QCD. Shown are the singular points, but not the cuts attached to each of them. Recall that $\beta_0 < 0$ according to Eq. (2.18).

['t Hooft 1977]

CAN WE MAKE SENSE OUT OF "QUANTUM CHROMODYNAMICS"?

G. 't Hooft

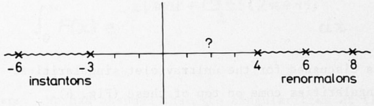
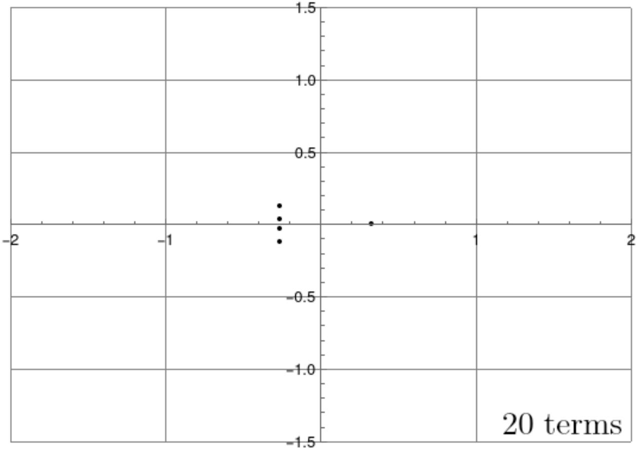


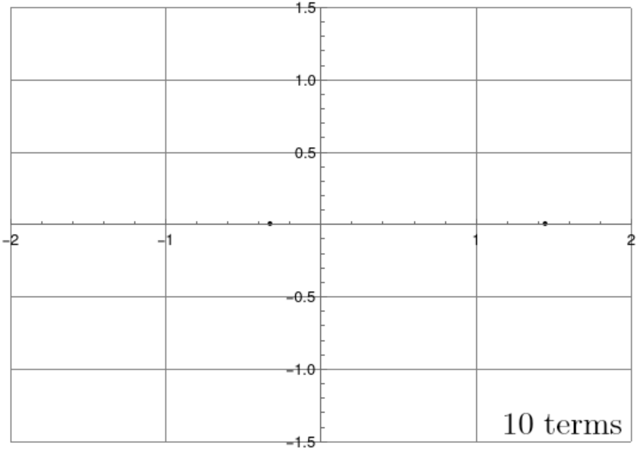
Fig. 4 Singularities in the Borel z variable for $\lambda\phi^4$. The units are $16\pi^2/3$. The question mark denotes the singularity that may be cancelled according to Parisi's mechanism.

Borel plane of tropical beta function in MS to 400 loops



Singularities at multiples of $-\frac{1}{3}$, no renormalons, essential singularity at ∞.

Tropical beta function in the MOM scheme



Still singularities at multiples of $-\frac{1}{3}$, but now additionally renormalons!

Nature of the leading singularity in MS

- Measure (this is harder than it looks) growth rate, find for $n \to \infty$

$$\beta_n^{(\mathrm{MS})} \sim 1.0858867 \cdot (-3)^n \Gamma\left(n+\frac{5}{2}\right)\left(1 - \frac{3.021658}{n^{1/3}} + \frac{3.35}{n^{2/3}} + \frac{0.94\log(n)}{n} - \frac{1.1}{n}\dots\right),,$$

- Factor 3^n compared to 1^n in conventional ϕ_4^4 theory.
- This has corrections $\frac{1}{\sqrt[3]{n}}$, not just $\frac{1}{n}$. There are also $\log(n), \log(n)^2, \dots$.

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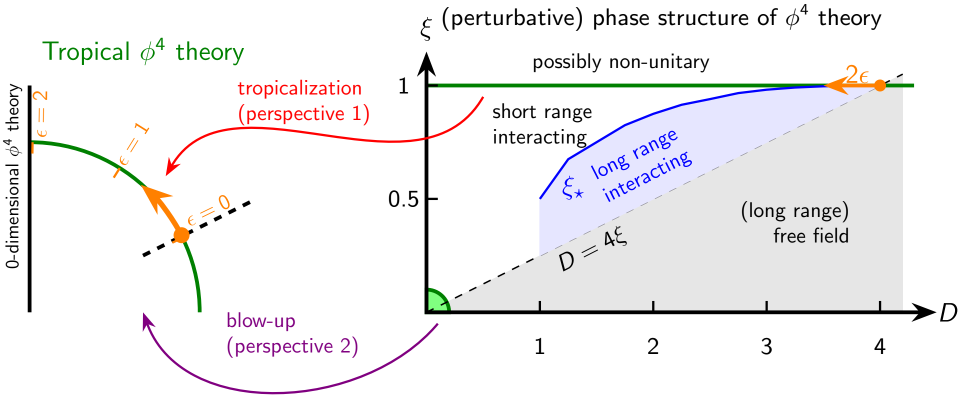
- Factor 3^n compared to 1^n in conventional ϕ_4^4 theory.
- This has corrections $\frac{1}{\sqrt[3]{n}}$, not just $\frac{1}{n}$. There are also $\log(n), \log(n)^2, \dots$.
- $\Leftrightarrow$ the leading singularity is at $u = -\frac{1}{3}$, is 6-fold confluent algebraic with exponents $b + \frac{i}{3}$ for $i \in \{0, \dots, 5\}$, where $b = -\frac{N+4}{2}$.
- This is more complicated than what is discussed in existing ϕ_4^4 asymptotics literature. We don't know if this structure is an artifact of tropicalization.

Conclusion

- Have introduced *tropical field theory* from 3 perspectives:
  1. As an approximation scheme for parametric Feynman integrals,
  2. As a special case of long-range ϕ^4 theory,
  3. (in the static case) as a perturbation expansion of mean field theory.



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  3. (in the static case) as a perturbation expansion of mean field theory.
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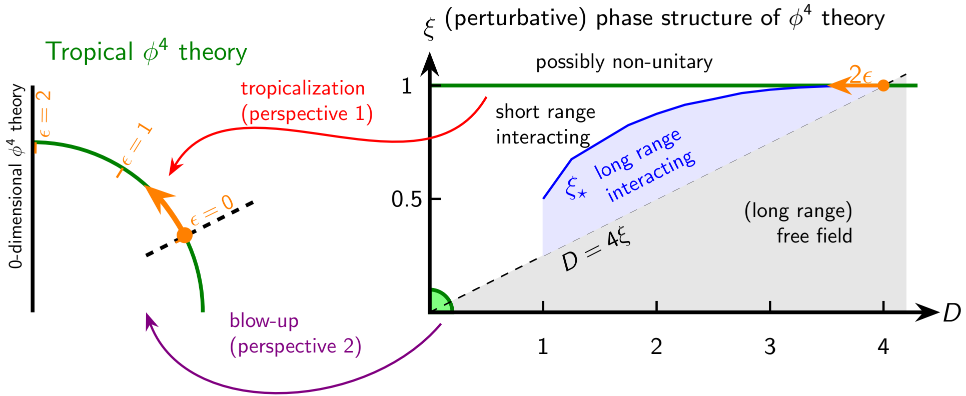
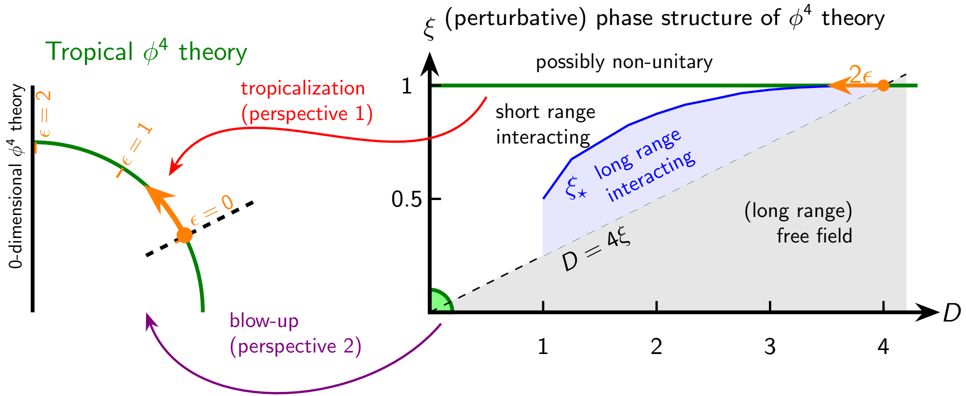


Conclusion

- Have introduced *tropical field theory* from 3 perspectives:
  1. As an approximation scheme for parametric Feynman integrals,
  2. As a special case of long-range ϕ^4 theory,
  3. (in the static case) as a perturbation expansion of mean field theory.
- Renormalization of coupling and mass works as usual, field strength is not renormalized.
- Borel plane of beta function has rich structure. Renormalons in MOM but not in MS. Fractional large-order asymptotics, confluent singularity, log corrections.



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Example: Renormalization at first orders, for $N = 1$

- For brevity, only show order $[\kappa^0]$. Bare 2-point and 4-point functions start with

$$[\kappa^0]G_4 = -\lambda + \frac{3}{\epsilon}\lambda^2 - \left(\frac{9}{\epsilon^2} + \frac{9}{\epsilon} - \frac{27}{2} + \dots\right)\lambda^3 + \left(\frac{27}{\epsilon^3} + \frac{72}{\epsilon^2} - \frac{69}{2\epsilon} - \frac{741}{4} + \dots\right)\lambda^4 + \dots$$

$$[\kappa^0]G_2 = 1 - \left(\frac{1}{2} + \dots\right)\lambda + \left(\underbrace{\frac{3}{2\epsilon}}_{\text{vertex subdivergence}} + \frac{5}{2} + \dots\right)\lambda^2 - \left(\underbrace{\frac{9}{2\epsilon^2} + \frac{15}{\epsilon}}_{\text{vertex subdivergence}} + \frac{227}{8} + \dots\right)\lambda^3 + \dots$$

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- Vertex renormalization $\lambda = \mu_0^{2\epsilon} g \cdot Z_g(g,\epsilon)$. Here: minimal subtraction (MS) scheme, Z includes only the poles in ϵ;

$$Z_g = 1 + \frac{3}{\epsilon}g + \left(\frac{9}{\epsilon^2}-\frac{9}{\epsilon}\right)g^2 + \left(\frac{27}{\epsilon^3}-\frac{63}{\epsilon^2}+\frac{87}{\epsilon}\right)g^3 + \left(\frac{81}{\epsilon^4}-\frac{621}{2\epsilon^3}+\frac{774}{\epsilon^2}-\frac{1407}{\epsilon}\right)g^4+\dots$$

- Logarithmic momentum scale $z = e^{-\epsilon L} = \left(\frac{\mu^2}{\mu_0^2}\right)^{-\epsilon}$, renormalized function

$$G_{\mathcal{R},4} = -gz + \left(-\frac{3}{\epsilon}z^2+\frac{3}{\epsilon}z\right)g^2 - \left(\left(\frac{9}{\epsilon^2}+\frac{9}{\epsilon}-\frac{27}{2}\pm\dots\right)z^3 - \frac{18}{\epsilon^2}z^2 + \left(\frac{9}{\epsilon^2}-\frac{9}{\epsilon}\pm\dots\right)z\right)g^3+\dots$$

$$= -g + \left(3L - \frac{3}{2}\epsilon L^2+\dots\right)g^2 + \left(\frac{9(-3-4L+2L^2)}{2} + 9\epsilon(1+3L+2L^2-L^3)+\dots\right)g^2+\dots$$

Critical exponents

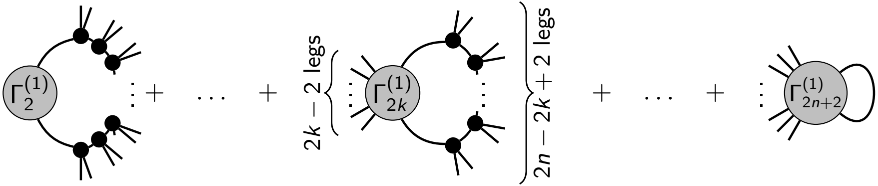
β depends on renormalization scheme, but not the correction to scaling exponent $\psi := \partial_g \beta(g_\star(\epsilon))$, where $g_\star$ is such that $\beta(g_\star(\epsilon), \epsilon) = 0$.
In the tropical theory:

$$g_\star = \frac{\epsilon}{3} + \frac{2\epsilon^2}{3} - \frac{5\epsilon^3}{9} + \frac{346\epsilon^4}{81} - \frac{22367\epsilon^5}{648} + \frac{412672\epsilon^6}{1215} - \frac{669955249\epsilon^7}{174960} + \frac{11743752875\epsilon^8}{244944} + \dots$$
$$\psi = 2\epsilon - 4\epsilon^2 + \frac{68\epsilon^3}{3} - \frac{1688\epsilon^4}{9} + 1897\epsilon^5 - \frac{3555593\epsilon^6}{162} + \frac{273066547\epsilon^7}{972} - \frac{842771759\epsilon^8}{216} + \dots.$$

In non-tropical case:

$$\psi \approx 2\epsilon - 2.518\epsilon^2 + 12.95\epsilon^3 - 83.76\epsilon^4 + 664.0\epsilon^5 - 5959\epsilon^6 + \dots.$$

Example: Recurrence at 2 loops



- 2-loop 1PI amplitude $\Gamma_n^{(2)}$ is a cycle of vertices and exactly one 1-loop 1PI amplitude $\Gamma_{2k}^{(1)}$.

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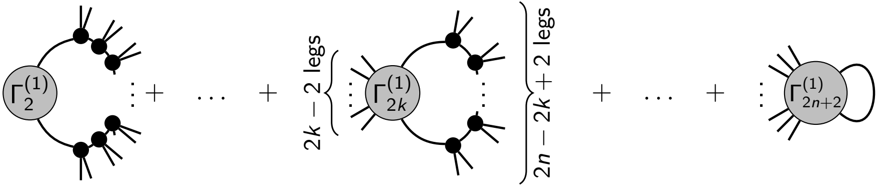

- 2-loop 1PI amplitude $\Gamma_n^{(2)}$ is a cycle of vertices and exactly one 1-loop 1PI amplitude $\Gamma_{2k}^{(1)}$.
- For each fixed k, have combinatorial factors
  - A factor $\Gamma_{2k}^{(1)}$ from the 1-loop amplitude which acts as an effective vertex.
  - A factor $(n+2-k)$ for the tropical integral of a ring of $(n+2-k)$ edges.
  - A symmetry factor consisting of
    - $(2n)!$ for the $2n$ external legs,
    - $\frac{1}{(2!)^{n+1-k}}$ for exchanging the pairs of external legs at the tree level vertices,
    - $\frac{1}{(2k-2)!}$ for permuting the $(2k-2)$ external legs of the effective vertex,
    - $\frac{1}{2}$ for reversing the direction of the cycle.
- End result: 2-loop amplitude is

$$\Gamma_{2n}^{(2)} = \frac{(2n)!}{2^{n+2}(2\epsilon+n-2)} \sum_{k=1}^{n+1} \frac{k(2k-1)(n+2-k)}{\epsilon+k-2}, \qquad n \geq 0.$$

Example: 2-loop 2-point amplitude

- The 2-loop 2-point function has $2n = 2$, so $n = 1$, and the summands $\frac{1}{2^{-k}(2k-2)!} \times (n+2-k) \times \Gamma^{(1)}_{2k}$ are:

$$\Gamma^{(2)}_2 = \frac{2}{8(2\epsilon-1)}\Big(\underbrace{2 \times 2 \times \frac{1}{2(\epsilon-1)}}_{\text{for } k=1} + \underbrace{2 \times 1 \times \frac{3}{\epsilon}}_{\text{for } k=2}\Big) = \frac{4\epsilon-3}{2\epsilon(\epsilon-1)(2\epsilon-1)}.$$

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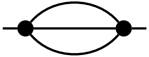
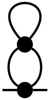
- Corresponding Feynman graphs: Sunrise and double tadpole

$S :=$  , $P :=$  .

- Use combinatorial formula:

$$\mathcal{I}_{\text{tr}}[S] = \frac{1}{2\epsilon - 1}\left(\frac{2}{\epsilon} + \frac{2}{\epsilon} + \frac{2}{\epsilon}\right) = \frac{6}{\epsilon(2\epsilon - 1)}, \qquad \mathcal{I}_{\text{tr}}[P] = \mathcal{I}_{\text{tr}}[r_1] \cdot \mathcal{I}_{\text{tr}}[r_2] = \frac{1}{\epsilon - 1} \cdot \frac{2}{\epsilon}$$

- Including symmetry factors, we reproduce

$$\Gamma^{(2)}_2 = \frac{1}{6} \times \frac{6}{\epsilon(2\epsilon - 1)} + \frac{1}{4} \times \frac{2}{\epsilon} \times \frac{1}{\epsilon - 1} = \frac{4\epsilon - 3}{2\epsilon(\epsilon - 1)(2\epsilon - 1)}.$$

Recurrence, general case

- For $\ell > 2$ loops additional freedom to distribute $(\ell - 1)$ loops over the vertices in the cycle $\Rightarrow$ Bell polynomial. Use variable t to count loop order,

$$\Gamma_{2n}^{(\ell)} = \frac{(2n)!}{2(\ell\epsilon - 2 + n)} \left[t^{\ell-1} \right] \sum_{k=1}^{\ell-1+n} \frac{k!}{(n+k)!} B_{n+k,k}\left(\Gamma_2,\ \Gamma_4,\ \frac{\Gamma_6}{4},\ \frac{\Gamma_8}{30},\ \ldots,\ \frac{(n+1)!\Gamma_{2n+2}}{(2n)!} \right).$$

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- Allow for 2-valent vertices with amplitude κ ($\Rightarrow$ geometric sum), and $O(N)$ symmetry ($\Rightarrow$ cycle either produces one or zero powers of N),

$$\frac{\Gamma_{2n,s}^{(\ell)}}{(2n)!} = \frac{1}{2(\ell\epsilon + n + s - 2)} \left[x^{2n} \cdot t^{\ell-1} \cdot \kappa^s\right] \left(\frac{1}{\left(1 - \sum_{j=0}^\infty x^{2j} \frac{\Gamma_{2j+2}}{(2j)!}\right)} + \frac{N-1}{\left(1 - \sum_{j=0}^\infty x^{2j} \frac{\Gamma_{2j+2}}{(2j+1)!}\right)} - N \right).$$

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- This is a PDE for the generating function $\mathcal{G}(x,t,\kappa) := \sum_{n=0}^\infty \sum_{\ell=0}^\infty \sum_{s=0}^\infty \Gamma_{n,s}^{(\ell)} \kappa^s t^\ell \frac{x^n}{n!}$ (see [Borinsky 2025] for $N = 1$, but arbitrary interactions)

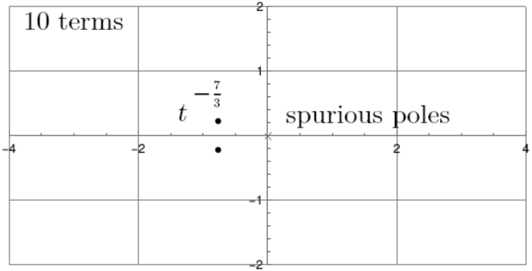
$$\left(2\epsilon\ t\partial_t + x\partial_x + 2\kappa\partial_\kappa - 4\right)\mathcal{G}(x,t,\kappa) = t \cdot \left(\frac{1}{1 - \partial_x^2 \mathcal{G}(x,t,\kappa)} + \frac{(N-1)x}{x - \partial_x \mathcal{G}(x,t,\kappa)} - N \right).$$

(interesting relation to 0-dimensional QFT, functional renormalization group, etc, but skip for now.)

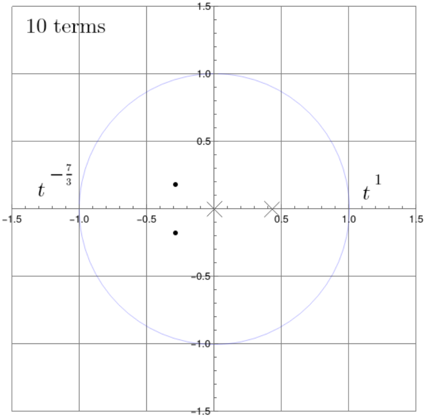
Example: Algebraic singularity

- Consider the function $\frac{t}{(1+t)^{\frac{7}{3}}}$ with increasingly many terms.
- Sequence of poles (dot) and zeros (crosses) accumulating at the branching point $t=-1 \in \mathbb{C}$.
- Singularity at $t=\infty$ is at $+1$ in the conformal disk.

Borel plane



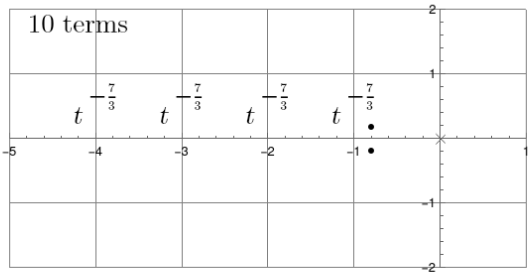
Mapped to unit disk



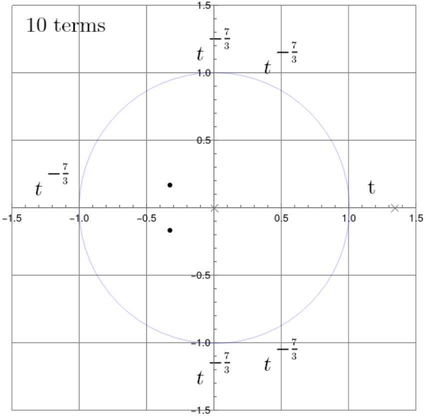
Example: Infinitely many algebraic singularities

- Now consider $\sum_{n=1}^{\infty} \frac{t}{(n+t)^{\frac{7}{3}}}$.
- We can resolve the first few singularities, more with increasing number of terms.
- Essential singularity at $t = -\infty$ is "shielded".

Borel plane



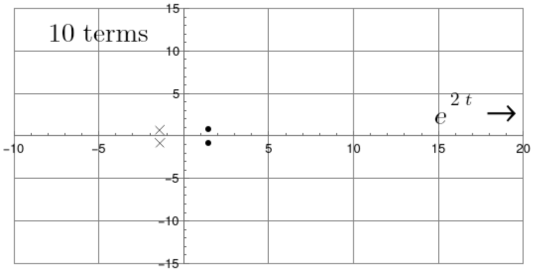
Mapped to unit disk



Example: Essential singularity at infinity

- The function e^{2t} has an essential singularity.
- Is represented by a "wall" of poles.

Borel plane



Mapped to unit disk